Gravitino Phenomenology with the Spinor Helicity Formalism

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Outline

- Introduction and Motivation,
- The Spinor Helicity Formalism,
- Application,
- Final Comments.

Motivation

"Scattering experiments are crucial to understand the fundamental blocks in nature".
The Standard Model of elementary particles was developed largely because scattering experiments, (the discovery the gauge bosons $W^{\pm}$ y Z^{0} and the gluons and quarks and more recently the Higgs bosons).

Motivation

The principal physical observable in the scattering experiments is the differential cross section (DCS) $\frac{d\sigma}{d\Omega}$.
Interpretation of DATA from scattering experiments is based mainly in the theoretical predictions of the cross section which are computed with **QFT** methods.
The DCS that connects theory with experiments is proportional to the modulus squared of the scattering amplitude

$$\boxed{\frac{d\sigma}{d\Omega} \propto |\mathcal{A}|^2.}$$

Motivation

There are several computer programs that undoubtedly help computing the scattering amplitudes (*MadGraph*, *Form*, *FeynRules*, *FeynCalc*, among others), most of them numerically.
For example to analytically compute $\tilde{t} \to b\ W\ G$ (with the LSP 3/2-spin gravitino particle):

$\tilde{t}_1 \to b + W + \tilde{\Psi}_\mu$

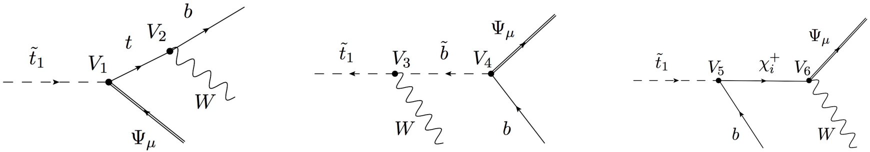
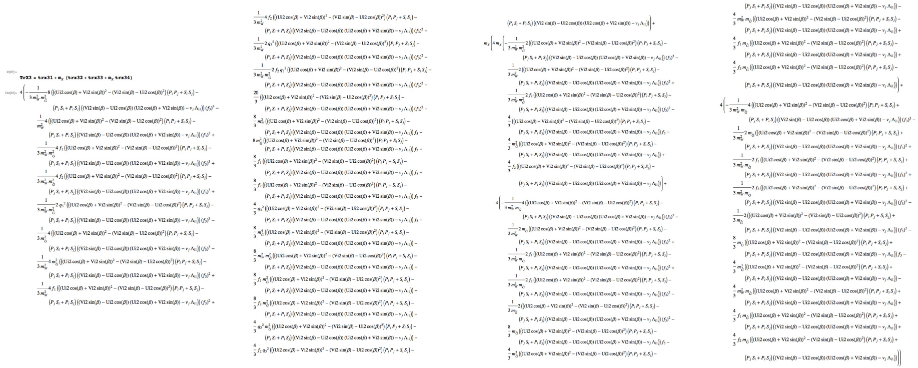


Figure 1. top diagram **Figure 2.** sbotom diagram **Figure 3.** chargino diagram

Lorenzo Diaz-Cruz & Bryan Larios-Lopez, EPJC (2016) ***76****:157.*

Considering just one channel and using Mathematica one obtain for the chargino $|\overline{\mathcal{M}_{1\chi}}|^2$.



Spinor Helicity Formalism (a pragmatic point of view)

Massless Case (*textbooks*)

The SHF is based in the following observation:

Fields with spin-1 transform in the $(\frac{1}{2}, \frac{1}{2})$ representation of the Lorentz group.

So we are able to express the 4-momentum of any particle as a biespinor: $p_\mu \rightarrow p_{a\dot{a}}$

$$\boxed{p_{a\dot{a}} = p_\mu \sigma^\mu_{a\dot{a}}} \qquad (1)$$

$$u_-(\vec{p}) = \begin{pmatrix} \phi_a \\ 0 \end{pmatrix}$$

$$u_-(\vec{p})\bar{u}_-(\vec{p}) = \frac{1}{2}(1-\gamma_5)(-\not{p}) = \begin{pmatrix} 0 & -p_{a\dot{a}} \\ 0 & 0 \end{pmatrix} \quad \boxed{p_{a\dot{a}} = -\phi_a \phi^*_{\dot{a}}} \qquad (2)$$

The key of the SHF is to considerate ϕ_a as the fundamental object and express the momenta of the particles in terms of ϕ_a.

Highly convenient and powerful notation

If p and k are the momenta and ϕ_a, κ_a their associated spinors, we can define the following products of spinors:

$$\boxed{[pk] = \phi^a \kappa_a = \bar{u}_+(\vec{p})u_-(\vec{k}) = -[kp]} \tag{3}$$

$$\boxed{\langle pk \rangle = \phi^*_{\dot{a}} \kappa^{*\dot{a}} = \bar{u}_-(\vec{p})u_+(\vec{k}) = -\langle kp \rangle} \tag{4}$$

Massless Dirac Spinor and Polarization vectors

$$|p] = u_-(\mathbf{p}) = v_+(\mathbf{p}) \; ,$$

$$|p\rangle = u_+(\mathbf{p}) = v_-(\mathbf{p}) \; ,$$

$$[p| = \overline{u}_+(\mathbf{p}) = \overline{v}_-(\mathbf{p}) \; ,$$

$$\langle p| = \overline{u}_-(\mathbf{p}) = \overline{v}_+(\mathbf{p}) \; .$$

$$\varepsilon_+^\mu(k) = -\frac{\langle q|\gamma^\mu|k]}{\sqrt{2}\,\langle q\,k\rangle} \; ,$$

$$\varepsilon_-^\mu(k) = -\frac{[q|\gamma^\mu|k\rangle}{\sqrt{2}\,[q\,k]} \; ,$$

With the spinor products

$$\bar{u}_-(\vec{p})\,u_-(\vec{k}) = 0 \qquad (5)$$

$$\bar{u}_+(\vec{p})\,u_+(\vec{k}) = 0 \qquad (6)$$

$$\boxed{\langle pk] = [pk\rangle = 0} \qquad (7)$$

Some properties of spinors products that are needed to compute scattering amplitudes are the following:

$$[ij] = -[ji],$$

$$\langle ij\rangle = [ji]^*,$$

$$\langle ij\rangle[ji] = \langle ij\rangle\langle ij\rangle^* = |\langle ij\rangle|^2,$$

$$\langle ij\rangle[ji] = -2k_i \cdot k_j = s_{ij},$$

$$\langle i|\gamma_\mu|j] = [j|\gamma_\mu|i\rangle,$$

$$\langle i|\gamma_\mu|j]\langle k|\gamma^\mu|l] = 2\langle ik\rangle[lj],$$

$$\langle ab\rangle\langle cd\rangle = \langle ac\rangle\langle bd\rangle + \langle ad\rangle\langle cb\rangle,$$

$$\sum_{k=1}^{n}\langle ik\rangle[kj] = 0,$$

What about massive particles ?

Light Cone Decomposition (LCD)

Let $\mathbf{p}^{\mu}$ be any time-like 4-momentum, we can decompose it into 2 light-like four momenta as follows.

$$\boxed{p^{\mu} \equiv r^{\mu} + \alpha q^{\mu}} \, . \tag{8}$$

$$u_{-} = |r] + \frac{m}{\langle rq\rangle}|q\rangle \quad , \quad u_{+} = \frac{m}{[rq]}|q] + |r\rangle;$$

$$v_{+} = |r] - \frac{m}{\langle rq\rangle}|q\rangle \quad , \quad v_{-} = -\frac{m}{[rq]}|q] + |r\rangle;$$

$$\bar{u}_{-} = \frac{m}{[qr]}[q| + \langle r| \quad , \quad \bar{u}_{+} = [r| + \frac{m}{\langle qr\rangle}\langle q|;$$

$$\bar{v}_{+} = -\frac{m}{[qr]}[q| + \langle r| \quad , \quad \bar{v}_{-} = [r| - \frac{m}{\langle qr\rangle}\langle q|.$$

$$\epsilon_{+}^{\mu} = \frac{\langle q|\gamma^{\mu}|r]}{\sqrt{2}\langle rq\rangle},$$

$$\epsilon_{-}^{\mu} = \frac{\langle r|\gamma^{\mu}|q]}{\sqrt{2}[qr]},$$

$$\epsilon_{0}^{\mu} = \frac{1}{2m}\left(\langle r|\gamma^{\mu}|r] - \alpha\langle q|\gamma^{\mu}|q]\right) = \frac{1}{m}r^{\mu} + \frac{m}{2p\cdot q}q^{\mu}.$$

Computing Scattering Amplitudes

Gravitino wave functions

We will consider a model in a local supersymmetric field theory ($\mathcal{N} = 1$ SUGRA) where the gravitino is the lightest supersymmetric particle (LSP), therefore a good DM candidate.

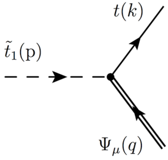
$$\tilde{\Psi}^{\mu}_{++}(p) = \epsilon^{\mu}_{+}(p)u_{+}(p),$$

$$\tilde{\Psi}^{\mu}_{--}(p) = \epsilon^{\mu}_{-}(p)u_{-}(p)$$

$$\tilde{\Psi}^{\mu}_{+}(p) = \sqrt{\frac{2}{3}}\epsilon^{\mu}_{0}(p)u_{+}(p) + \frac{1}{\sqrt{3}}\epsilon^{\mu}_{+}(p)u_{-}(p),$$

$$\tilde{\Psi}^{\mu}_{-}(p) = \sqrt{\frac{2}{3}}\epsilon^{\mu}_{0}(p)u_{-}(p) + \frac{1}{\sqrt{3}}\epsilon^{\mu}_{-}(p)u_{+}(p).$$

Two body NLSP Stop decay ($\tilde{t} \to t\,\tilde{G}$)



$$\boxed{\mathcal{M}_{\lambda_2\lambda_3} = -\frac{1}{\sqrt{2}M}\bar{\psi}^{\mu}_{\lambda_2}(p_2)\gamma_\alpha\gamma_\mu p_1^\alpha \hat{P}_R u_{\lambda_3}(p_3)} \tag{9}$$

$\lambda_2\lambda_3$	$\mathcal{M}_{\lambda_2\lambda_3}$
$-+$	$-\frac{\langle r_2 q_2\rangle}{\sqrt{3}M m_{\tilde{G}} s_{r_2 q_2}}\left(s^2_{r_2 q_2} - m^2_{\tilde{G}} m^2_t\right)$
$+-$	$-\frac{[r_2 q_2] m_t}{\sqrt{3}M s^2_{r_2 q_2}}\left(s^2_{r_2 q_2} - m^2_{\tilde{G}} m^2_t\right)$

Squaring the helicity amplitudes of the last Table, we obtain the following expression for the total squared amplitud

$$\langle|\mathcal{M}|^2\rangle = |\mathcal{M}_{-+}|^2 + |\mathcal{M}_{+-}|^2, \tag{10}$$

$$= \frac{\left(s_{r_2q_2}^4 - (m_t m_{\tilde{G}})^4\right)\left(s_{r_2q_2}^2 - (m_t m_{\tilde{G}})^2\right)}{3M^2 m_{\tilde{G}}^2 s_{r_2q_2}^3}, \tag{11}$$

$$= \frac{(m_{\tilde{t}}^2 - m_{\tilde{G}}^2 - m_t^2)\left((m_{\tilde{t}}^2 - m_t^2 - m_{\tilde{G}}^2)^2 - 4m_t^2 m_{\tilde{G}}^2\right)}{3M^2 m_{\tilde{G}}^2}, \tag{12}$$

To appreciate the power and efficiency of the SHF, we can remember the completeness relation for the spin-3/2 Gravitino Field that is needed in the traditional approach.

$$\sum_{\tilde{\lambda}=1}^{3} \Psi_\mu(\vec{p}_1, \tilde{\lambda})\overline{\Psi}_\nu(\vec{p}_1, \tilde{\lambda}) = -(\not{p}_1 + m_{\tilde{G}}) \times \left\{ \left(g_{\mu\nu} - \frac{p_\mu p_\nu}{m_{\tilde{G}}^2} \right) - \frac{1}{3} \left(g_{\mu\sigma} - \frac{p_\mu p_\sigma}{m_{\tilde{G}}^2} \right) \left(g_{\nu\lambda} - \frac{p_\nu p_\lambda}{m_{\tilde{G}}^2} \right) \gamma^\sigma \gamma^\lambda \right\}$$

And we still need to take into account other fields to compute the trace.

$e^+e^- \to \gamma\,\tilde{G}\,\tilde{G}$

Using the SUSY QED model constructed by Mawatari and Oexl (arXiv:1402.3223v2), and applying the **SHF** we will compute the scattering amplitude for the $e^+e^- \to \gamma\,\tilde{G}\,\tilde{G}$ reaction. It is important to mention that the cross sections for this reaction has been computed numerically, so it is interesting have an analytical result.

Feynman Diagrams

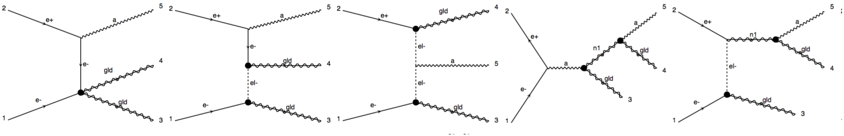


Figure: Feynman Diagrams for $e^+e^- \to \gamma\,\tilde{G}\,\tilde{G}$

$e^+e^- \to \gamma\tilde{G}\tilde{G}$

We have 5 Feynman diagrams each on them with 5 external (massless) particles, each particle have two helicity states $(\pm)$, in principle we need to compute 2^5 helicity amplitudes for each diagram.

Feynman Diagrams

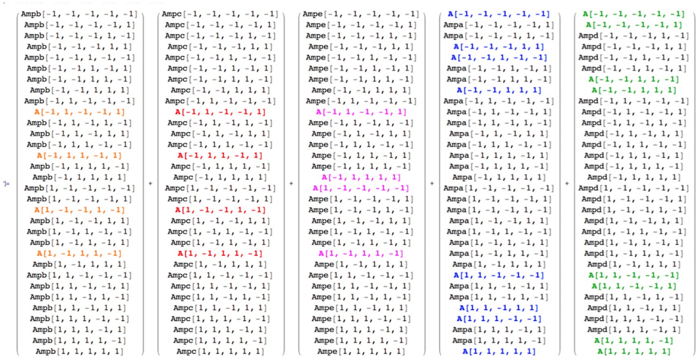


Figure: All the helicity amplitudes

$e^+e^- \to \gamma \tilde{G}\tilde{G}$

One of the several and marvelous advantages of the helicity amplitudes is that it is easy to identify the symmetries as well as the null helicity amplitudes. We already know that the terms $\langle xy]$ and $[xy\rangle$ are zero, a small program could help us to find which helicity amplitude is zero.

$$\mathcal{A}^a_{\lambda_1\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5} \approx \left(\bar{v}_{\lambda_2}(2)\not{\epsilon}_{\lambda_3}(3)\not{q}u_{\lambda_1}(1)\bar{u}_{\lambda_5}(5)v_{\lambda_4}(4)\right), \tag{13}$$

$$\mathcal{A}^a_{-+--+} \approx \left(\bar{v}_+(2)\not{\epsilon}_-(3)\not{q}u_-(1)\bar{u}_+(5)v_-(4)\right), \tag{14}$$

$$\mathcal{A}^a_{-+--+} \approx [54\rangle = 0. \tag{15}$$

Counting helicity amplitudes

$$
\begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\ A[-1,1,-1,-1,1]\\0\\0\\0\\ A[-1,1,1,-1,1]\\0\\0\\0\\0\\ A[1,-1,-1,1,-1]\\0\\0\\0\\ A[1,-1,1,1,-1]\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix}
+
\begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\ A[-1,1,-1,-1,1]\\0\\0\\0\\0\\ A[-1,1,1,1,-1]\\0\\0\\ A[1,-1,-1,-1,1]\\0\\0\\0\\0\\ A[1,-1,1,1,-1]\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix}
+
\begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\ A[-1,1,-1,-1,1]\\0\\0\\0\\ A[-1,1,1,-1,1]\\0\\0\\0\\0\\ A[1,-1,-1,1,-1]\\0\\0\\0\\ A[1,-1,1,1,-1]\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix}
+
\begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\ A[-1,1,-1,-1,1]\\0\\0\\0\\ A[-1,1,1,-1,1]\\0\\0\\0\\0\\ A[1,-1,-1,1,-1]\\0\\0\\0\\ A[1,-1,1,1,-1]\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix}
+
\begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\ A[-1,1,-1,-1,1]\\0\\0\\0\\0\\0\\ A[-1,1,1,1,1]\\ A[1,-1,-1,-1,-1]\\0\\0\\0\\0\\0\\ A[1,-1,1,1,-1]\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix}
$$

We started our problem with **160** helicity amplitudes, but just looking at the possible helicity states of the external particles we found that there are only **20** helicity amplitudes to compute. Applying complex conjugation we really need to compute half of the final helicity amplitudes. *At the end, just the 6% of the work will be done and without the help of any machine if one desires*.

The total modulus squared amplitude is as follows:

$$
\begin{aligned}
|\mathcal{M}|^2 &= \sum_{perm} |A_{\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5}|^2 \\
&= 2\Big(|\mathcal{A}_{-++-+}|^2 + |\mathcal{A}_{-+++-}|^2 + |\mathcal{A}_{-++++}|^2 + |\mathcal{A}_{-+--+}|^2\Big)
\end{aligned}
$$

where each squared helicity amplitude takes the form:

$$|\mathcal{A}_{-++-+}|^2 = \frac{16C^2|s_{24}|^2|s_{15}||s_{34}|}{[(z^2 - m^2_{\tilde{e}_-})(r^2 - m^2_{\tilde{e}_-})]^2|s_{23}|} \tag{16}$$

$$|\mathcal{A}_{-+++-}|^2 = \frac{128D^2|s_{34}||s_{25}||s_{12}||s_{24}|}{[q'^2(r'^2 - m^2_{\chi_O})]^2} \tag{17}$$

$$|\mathcal{A}_{-++++}|^2 = \frac{8E^2|s_{34}|^2|s_{24}||s_{15}|}{[(l'^2 - m^2_{\chi_0})(x^2 - m^2_{\tilde{e}_-})]^2} \tag{18}$$

$$|\mathcal{A}_{-+--+}|^2 = |s_{23}||s_{34}||s_{51}|\xi_1^2 + |s_{34}|^2|s_{45}||s_{51}|\xi_2^2 \tag{19}$$
$$+ 2Re[\xi_1\xi_2|s_{51}||s_{34}|[23]|\langle 34\rangle[45]\langle 52\rangle]$$

Conclusions

- We compute **analytically** the total scattering amplitude for the reaction $e^+e^- \rightarrow \gamma\tilde{G}\tilde{G}$.
- It was shown that the **SHF** is a powerful method, in fact it is much more economic than the traditional approach.
- With the complete result ($e^+e^- \rightarrow \gamma\tilde{G}\tilde{G}$), it is possible to compare the cross section with the numerical results.
- From this point, we can now start to do physics.

Thank you

Algunas fuentes Bibliogrficas:

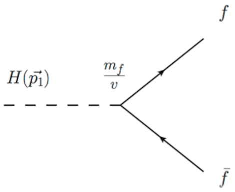
1. H. Elvang, Y.T. Huang, “Scattering Amplitudes in gauge and Gravity”, Cambridge **2015**.
2. M. Srednicki, “QFT”, Chap. 50, Cambridge **2009**.
3. M. Schwartz, “QFT and SM”, Chap. 27, Cambridge **2014**.
4. L. Daz Cruz, B. Larios, O.Meza, J.Perez “Espinores de Weyl y el Formalismo de Helicidad”, Rev. Mex. Fis. E **61(2) (2015)** 104.

In spite of the tremendous difficulties lying ahead, I feel that S-matrix theory is far from dead and that . . . much new interesting mathematics will be created by attempting to formalize it.

"Tullio Regge"

2-body Higgs decay $h(p_1) \to f(p_2)\bar{f}(p_3)$

From the Feynman diagram we get the helicity amplitudes (HAs) (as is usual)



$$\boxed{\mathcal{M}_{\lambda_2\lambda_3}(p_1,p_2,p_3) = \frac{m_f}{v}\bar{u}_{\lambda_2}(p_2)v_{\lambda_3}(p_3),} \tag{20}$$

$\lambda_2\lambda_3$	$\mathcal{M}_{\lambda_2\lambda_3}$
$--$	$\frac{m_f}{v[q_2q_3]}(s_{q_2q_3} - m_f^2)$
$++$	$\frac{m_f}{v\langle q_2q_3\rangle}(s_{q_2q_3} - m_f^2)$

2-body Higgs decay $h(p_1) \to f(p_2)\bar{f}(p_3)$

Let me compute these HAs step by step

- $\bar{u}_-(p_2),\ \bar{u}_+(p_2)$

$$\mathcal{M}_{\lambda_2\lambda_3}(p_1,p_2,p_3) = \frac{m_f}{v}\,\bar{u}_{\lambda_2}(p_2)\,v_{\lambda_3}(p_3) \tag{21}$$

- $v_-(p_3),\ v_+(p_3)$

We see that there are 4 HAs, these are $-+, --, +-++$ (In the massless case there are just 2, +- and -+).

2-body Higgs decay $h(p_1) \to f(p_2)\bar{f}(p_3)$

$$\mathcal{M}_{--} = \frac{m_f}{v}\bar{u}_-(2)v_-(3) \tag{22}$$

$$= \frac{m_f}{v}\left(\frac{m_f}{[q_2 r_2]}[q_2| + \langle r_2|\right)\left(-\frac{m_f}{[r_3 q_3]}|q_3] + |r_3\rangle\right) \tag{23}$$

$$= \frac{m_f}{v}\left(-\frac{m_f^2[q_2 q_3]}{[q_2 r_2][r_3 q_3]} + \frac{m_f}{[q_2 r_2]}\cancelto{0}{[q_2 r_3\rangle} - \frac{m_f\cancelto{0}{\langle r_2 q_3]}}{[r_3 q_3]} + \langle r_2 r_3\rangle\right) \tag{24}$$

$$= \frac{m_f}{v}\left(-\frac{m_f^2[q_2 q_3]}{[q_2 r_2][r_3 q_3]} + \langle r_2 r_3\rangle\right) \tag{25}$$

For this special example we will use simultaneous LCD (SLCD),

$$p_2 = r_2 - \frac{m_f^2}{2r_2 \cdot q_2} q_2 \tag{26}$$

$$p_3 = r_3 - \frac{m_f^2}{2r_3 \cdot q_3} q_3 \tag{27}$$

For this special example we will use simultaneous LCD (SLCD),

$$p_2 = r_2 - \frac{m_f^2}{2 r_2 \cdot q_2} q_2 \tag{28}$$

$$p_3 = q_2 - \frac{m_f^2}{2 q_2 \cdot r_2} r_2 \tag{29}$$

We choose $r_3 = q_2$ and $q_3 = r_2$, this will reduce our calculations.

2-body Higgs decay $h(p_1) \to f(p_2)\bar{f}(p_3)$

Returning to the HA $\mathcal{M}_{--}$ and taking into account SLCD, we have

$$\mathcal{M}_{--} = \frac{m_f}{v}\left(-\frac{m_f^2\cancel{[q_2r_2]}}{\cancel{[q_2r_2]}[q_2r_2]} + \langle r_2q_2\rangle\right) \tag{30}$$

$$= \frac{m_f}{v}\left(-\frac{m_f^2}{[q_2r_2]} + \langle r_2q_2\rangle\right) = \frac{m_f}{v[q_2r_2]}(-m_f^2 + \langle r_2q_2\rangle[q_2r_2]) \tag{31}$$

$$= \frac{m_f}{v[q_2r_2]}(-m_f^2 + s_{q_2r_2}) \tag{32}$$

We do not need to compute $\mathcal{M}_{++}$, we just complex conjugate $\mathcal{M}_{--}$ (one of the greatest advantages of the SHF).

$$\mathcal{M}_{++} = \frac{m_f}{v\langle q_2 r_2\rangle}(-m_f^2 + s_{q_2 r_2}) \tag{33}$$

As you can guess we shall compute just one HA ($\mathcal{M}_{-+}$)

$$\mathcal{M}_{-+} = \frac{m_f}{v}\bar{u}_-(2)v_+(3) = \frac{m_f}{v}\left(\frac{m_f}{[q_2 r_2]}[q_2| + \langle r_2|\right)\left(|q_2] - \frac{m_f}{\langle r_2 q_2\rangle}|r_2\rangle\right) \tag{34}$$

$$\propto \cancelto{0}{[q_2 q_2]} - \cancelto{0}{[q_2 r_2\rangle} + \cancelto{0}{\langle r_2 q_2]} - \cancelto{0}{\langle r_2 r_2\rangle} \tag{35}$$

$$= 0 \tag{36}$$

Finally the averaged squared amplitude is then:

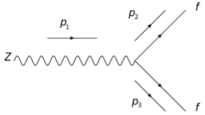
$$\boxed{\left\langle |\mathcal{M}|^2 \right\rangle = 2|\mathcal{M}_{--}|^2 = \frac{2m_f^2}{v^2 s_{q_2q_3}}(s_{q_2q_3} - m_f^2)^2 = \frac{y^2}{v^2}(1-4y^2),} \tag{37}$$

with $y = \frac{m_f}{M_h}$. Then the decay width Γ goes as follows

$$\boxed{\Gamma(h \to f\bar{f}) = \frac{\alpha_W M_h y^2}{8}(1-4y^2)^{3/2}.} \tag{38}$$

2-body Z boson decay $Z(p_1) \to f(p_2)\bar{f}(p_3)$

Again from the Feynman diagram we get the helicity amplitudes



$$\boxed{\mathcal{M} = \frac{1}{2} g_Z \epsilon_\mu(p_1)\bar{u}(p_2)\gamma^\mu(v_f - a_f\gamma^5)v(p_3)} \tag{39}$$

$\lambda_1\lambda_2\lambda_3$	$\mathcal{M}_{\lambda_1\lambda_2\lambda_3}$
$++-$	$\frac{1}{2}g_Z\frac{\langle p_2\vert\gamma_\mu\vert r_1]}{\sqrt{2}\langle r_1p_2\rangle}(v_f-a_f)[p_2\vert\gamma^\mu\vert p_3\rangle = \frac{g_Z(v_f-a_f)}{\sqrt{2}}\frac{\langle p_2p_3\rangle[r_1p_2]}{\langle r_1p_2\rangle}$
$0+-$	$\frac{1}{2}g_Z\left(\frac{1}{M}r_{1\mu}+\frac{M}{2p_{12}}p_{2\mu}\right)(v_f-a_f)[p_2\vert\gamma^\mu\vert p_3\rangle = \frac{g_Z(v_f-a_f)}{\sqrt{2}M}\langle r_1p_3\rangle[r_1p_2] = 0$
$-+-$	$\frac{1}{2}g_Z\frac{\langle r_1\vert\gamma_\mu\vert p_2]}{\sqrt{2}[p_2r_1]}(v_f-a_f)[p_2\vert\gamma^\mu\vert p_3\rangle = 0$

Finally the squared averaged scattering amplitude is as follows

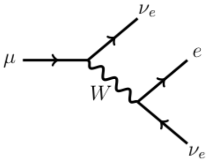
$$\boxed{\langle|\mathcal{M}|^2\rangle = \frac{1}{3}\left(|\mathcal{M}_{++-}|^2 + |\mathcal{M}_{--+}|^2\right) = \frac{g_Z^2 M^2}{3}\left(|v_f|^2 + |a_f|^2\right)} \tag{40}$$

The decay width for this channel is then:

$$\boxed{\Gamma\left(Z \to f\bar{f}\right) = \frac{g_Z^2 M}{48\pi}\left(|v_f|^2 + |a_f|^2\right).} \tag{41}$$

3-body Muon Decay $\mu(p_1) \to \bar{\nu}_{e^-}(p_2)\,\nu_\mu(p_3)\,e^-(p_4)$

From the Feynman diagram we get the helicity amplitudes (HAs) (as is usual)



$$\mathcal{M}_{\lambda_1\lambda_2\lambda_3\lambda_4} = \left(\frac{g_W}{\sqrt{8}M_W}\right)^2 [\bar{u}_{\lambda_3}(p_3)\gamma^\mu(1-\gamma_5)u_{\lambda_1}(p_1)]\,[\bar{u}_{\lambda_4}(p_4)\gamma_\mu(1-\gamma_5)v_{\lambda_2}(p_2)]\,,$$
$$= \left(\frac{g_W}{\sqrt{8}M_W}\right)^2 \mathcal{A}^\mu_{\lambda_3\lambda_1}\mathcal{B}_{\mu\lambda_4\lambda_2},$$

$\lambda_1\lambda_2\lambda_3\lambda_4$	$\mathcal{A}^{\mu\lambda_3\lambda_1}$	$\mathcal{B}_\mu^{\lambda_4\lambda_2}$	$\mathcal{A}^{\mu\lambda_3\lambda_1}\mathcal{B}_\mu^{\lambda_4\lambda_2}$	$\mathcal{M}\quad(p_2 = q_1,\, p_3 = q_4)$
$+-+-$	$2[p_3\vert\gamma^\mu\vert r_1\rangle$	$2[r_4\vert\gamma_\mu\vert p_2\rangle$	$4\langle p_2 r_1\rangle\langle p_3 r_4\rangle$	$\left(\frac{g_W}{\sqrt{2}m_\mu}\right)^2\langle p_2 r_1\rangle[p_3 r_4]$
$+++-$	$2[p_3\vert\gamma^\mu\vert r_1\rangle$	$\frac{2m_e}{[q_4 r_4]}[q_4\vert\gamma_\mu\vert p_2\rangle$	$4\frac{m_e}{[q_4 r_4]}\langle p_2 r_1\rangle\langle p_3 q_4\rangle$	0
$--++$	$\frac{2m_\mu}{\langle r_1 q_1\rangle}[p_3\vert\gamma^\mu\vert q_1\rangle$	$2[r_4\vert\gamma_\mu\vert p_2\rangle$	$4\frac{m_\mu}{\langle r_1 q_1\rangle}\langle p_2 q_1\rangle[p_3 r_4]$	0
$-++-$	$\frac{2m_\mu}{\langle r_1 q_1\rangle}[p_3\vert\gamma^\mu\vert q_1\rangle$	$\frac{2m_e}{[q_4 r_4]}[q_4\vert\gamma_\mu\vert p_2\rangle$	$4\frac{m_\mu m_e}{\langle r_1 q_1\rangle[q_4 r_4]}\langle p_2 q_1\rangle[p_3 q_4]$	0

The squared and averaged amplitude for the muon decay is:

$$\boxed{\langle|\mathcal{M}^{+-+-}|^2\rangle = \frac{1}{2}|\mathcal{M}^{+-+-}|^2 = 2\left(\frac{g_W}{M_W}\right)^4 (p_1\cdot p_2)(p_3\cdot p_4)} \tag{42}$$

From this result we can arrive to the decay width, which agrees with result of textbooks.