The Heisenberg model for high energy nucleon-nucleon scattering and AdS/CFT

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based on: (K. Kang and H.N. PLB 2005 and) H.N., J. Sonnenschein PRD 2015, H.N. JHEP 2016

Summary:

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0. Introduction

.

•At $s \to \infty$ (QCD) collision cross section is bound by unitarity (Froissart bound)

$$\sigma_{\text{tot}}(s) \leq C \ln^2 \frac{s}{s_0}, \quad C \leq \frac{\pi}{m_\pi^2}$$

•But 1952 Heisenberg model saturates bound (in QCD: hard!)

•Use AdS/CFT to look again at the Heisenberg model

•Asymptotic regime $\leftrightarrow$ scattering creating BHs in IR.

•Asymptotic regime: nucleon-nucleon (e.g. pp) collision at LHC/ heavy ion collisionss at RHIC/LHC.

- We will extend extend the Heisenberg model, show that it is quite unique

- We will extend to corrections to $\sigma_{\text{tot}}(s)$, as well as $\sigma_{\text{elastic}}(s)$.

- We will suggest an extension to be used for the sQGP hydrodynamics

- We will obtain "DBI scalar viscous hydrodynamics" for sQGP $\rightarrow$ restricted hydro model.

1. AdS/CFT and high energy scattering

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•String theory: all fundamental particles (matter and gauge bosons) are actually strings, living in $D = 10$ spacetime dimensions, with 16 or 32 supercharges.

•KK compactification to 3+1 dimensions + breaking susy to $\mathcal{N} = 1 \Rightarrow$ should get MSSM (susy version of Standard Model)

•But instead: AdS/CFT: model strongly coupled field theories via *duality* to string theory in some *curved* background (**not** real world)

•Standard example: $SU(N)$ $\mathcal{N} = 4$ SYM in 3+1 dimensions $\leftrightarrow$ string theory in $AdS_5 \times S^5$ background, for $N \to \infty$, $g^2N =$ fixed, large.

•But interested in high energy scattering in QCD $\rightarrow$ with Λ_{QCD}.

•Polchinski-Strassler (2001): "hard-wall model"

$$ds^2 = \frac{r^2}{R^2}d\vec{x}^2 + \frac{R^2}{r^2}dr^2 + R^2 ds_X^2, \quad r \leq r_{\text{min}} = R^2\Lambda_{\text{QCD}}.$$

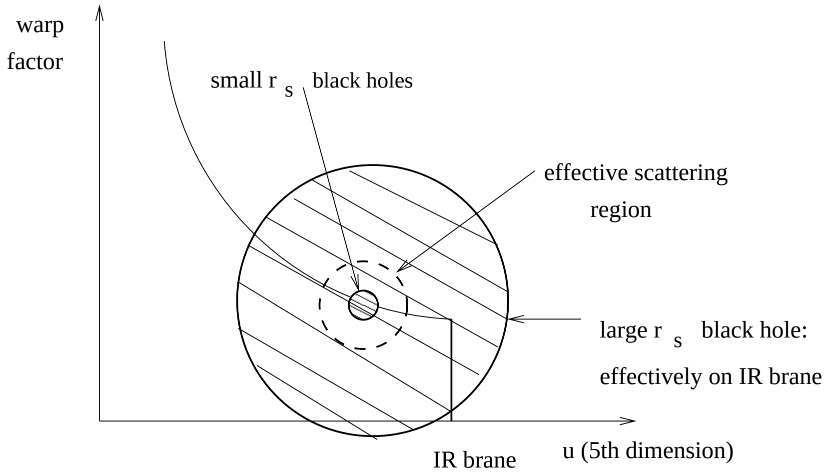
•Momenta rescaled: $\tilde{p}_{(AdS)} = \frac{R}{r}p_{(QCD)}$.

•"Glueball" wavefunction $e^{ip\cdot x} \leftrightarrow$ AdS wavefunction $e^{ip\cdot x}\psi(r,\Omega)$.

$$\mathcal{A}_{\text{QCD}}(p_i) = \int dr d^5\Omega \sqrt{g}\mathcal{A}_{\text{string}}(\tilde{p}_i)\prod_i \psi_i$$

•From $r \geq r_{\text{min}} \Rightarrow \sqrt{\alpha'}\tilde{p} \leq \sqrt{\hat{\alpha}'}p$, $\hat{\alpha}' = \sqrt{g_{YM}^2 N}\Lambda_{\text{QCD}}^{-2}$= QCD string tension.

•Find various regimes in QCD (HN, 2004 and 2005). Max behaviour: Froissart saturation $\leftrightarrow$ create black holes on the IR brane (cut-off).



Scattering in the cut-off AdS gravity dual. Small black holes are effectively produced in a scattering region and don't feel the IR brane. Asymptotically large black holes live effectively on the IR brane and have negligible quantum fluctuations in the 5th dimension.

•In the IR limit, BH $\simeq$ classical. Corresponds to sQGP fireball (HN, 2005): $\exists$ classical field theory description?

•Yes: Heisenberg's 1952 model.

•High energy gravitational scattering at $\sqrt{s} \sim M_{\mathsf{Pl}}$: gravitational shockwave scattering. (K. Kang and HN, 2005, HN 2005)

2. The original Heisenberg's model

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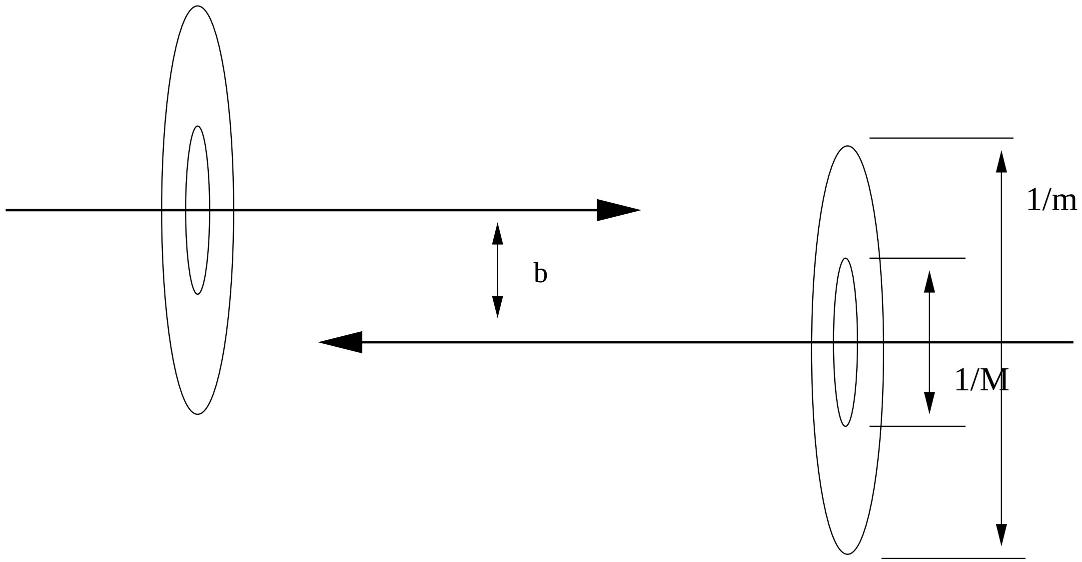
•Ideas: at $s \to \infty$, QCD scattering dominated by pion (lightest particle) *classical* field: large number of pions produced.

•But pion action is neither free, nor potential oriented (will see: it doesn't work) $\to$ is nonlinear in derivatives: massive DBI.

$$\mathcal{L} = l^{-4}\left[1 - \sqrt{1 + l^4[(\partial_\mu\phi)^2 + m^2\phi^2]}\right].$$

•Asymptotic: scattering of pion field shockwaves (Lorentz contraction: pancakes $\to$ shockwaves for both source and field)



Hadron scattering in the center of mass frame. M= hadron mass, m= pion mass. Also, A-S shockwave scattering on the IR brane. M= dual particle size. m=KK graviton mass (gravitational field in 5d, with given boundary conditions)

•Find $\sigma_{\text{tot}}(s)$ from a combination of classical and quantum: average emitted energy per pion $\langle k_0 \rangle = \mathcal{E}/n$ from classical scalar field + quantization (momentum $k \equiv$ pion energy k_0).

•Then, wavefunction at large r, $\psi(r) \sim e^{-m_\pi r}$ and energy loss $\mathcal{E} \propto \sqrt{s}$ and $\propto \psi(b) \sim e^{-m_\pi b}$. At b_{max}, emit *one* pion $\Rightarrow$

$$(\mathcal{E} =)\langle k_0 \rangle = \frac{\mathcal{E}}{n} = \sqrt{s} e^{-m_\pi b_{\text{max}}}$$

•Then, cross section = *classical* one,

$$\sigma_{\text{tot}}(s) = \pi b_{\text{max}}^2 = \frac{\pi}{m_\pi^2} \log^2 \frac{\sqrt{s}}{\langle s \rangle}.$$

•But only for DBI we find $\langle k_0 \rangle \simeq$ const. For free or $V = \sum_p \lambda_p \phi^p$, $\langle k_0 \rangle \propto \sqrt{s}$.

•Shockwave limit: $\phi = \phi(s)$, where $s = t^2 - x^2 \to$ boost invariant solution needed *to move on* $x^{\pm} = 0 \to$ restricts $\phi(x^+, x^-)$ to $\phi(x^+ x^-) = \phi(s)$.

•On $\phi = \phi(s)$, $(\partial_\mu \phi)^2 = -4s(d\phi/ds)^2$, and

$$\mathcal{L} = l^{-4}\left[1 - \sqrt{1 + l^4\left(-4s\left(\frac{d\phi}{ds}\right)^2 + m^2\phi^2\right)}\right]$$

•Solution

$$\phi = \frac{\sqrt{s}}{l^2}(1 + a\ s\ m^2 + ...)\ , \quad 0 \leq s \ll 1/m^2$$

•But need $(\partial_\mu \phi)^2$ finite at the shock $\Rightarrow \phi \sim A\sqrt{s}$ as $s \to 0$. Free theory: $\phi = A \log s/s_0 \Rightarrow (\partial_\mu \phi)^2 = -4A^2/s \to \infty$.

•Energy density:

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L} = \frac{l^{-4} + (\nabla\phi)^2 + m^2\phi^2}{\sqrt{1 + l^4[(\partial_\mu\phi)^2 + m^2\phi^2]}} - l^{-4}$$

.

•Perturbations: Square root finite, instead of zero. $\rightarrow \exists$ cut-off

•Find

$$\phi - l^{-2} i \frac{|t|}{|k|} \propto |t|^{1/2} |k|^{-3/2} e^{-i|k||t|} \Rightarrow$$

$$\frac{dE}{dk} \propto k^2 \phi(k)^2 \sim \frac{\text{const.}}{k} \quad \text{for} \quad m \leq k \leq m\gamma$$

and $k = k_0$ (pion energy), $E = \mathcal{E}$. One finally finds

$$\langle k_0 \rangle \equiv \frac{\mathcal{E}}{n} = m \frac{\ln\gamma}{1 - \frac{1}{\gamma}} \simeq \text{const.}$$

•But for $V = \sum_p \lambda_p \phi^p \Rightarrow$ find $d\mathcal{E}/dk_0 = B =$ const. $\Rightarrow$

$$\langle k_0 \rangle \propto \frac{\sqrt{\tilde{s}}}{\ln\sqrt{\tilde{s}}} \Rightarrow \text{not} \quad \text{good}$$

3. AdS/CFT description of saturation of Froissart bound

.

•High enough energy ⇒ scattering of Aichelburg-Sexl gravitational shockwaves (BHs boosted to $v = c$)

$$\begin{aligned} ds^2 &= 2dx^+ dx^- + (dx^+)^2 \Phi(x^i)\delta(x^+) + d\vec{x}^2 \\ \Delta_{D-2}\Phi(x^i) &= -16\pi G p \delta^{D-2}(x^i) \end{aligned}$$

•When

$$R > M_{\rm Pl} = g_s^{-1/4}\alpha'^{-1/2} = g_s^{-1/4}(R^2/\sqrt{g_s N})^{-1/2} = N^{1/4} R$$
$$\leftrightarrow N^{1/4}\Lambda_{\rm QCD} \equiv \hat{M}_{\rm Pl}$$

2 shockwaves collide, producing BHs (Giddings, 2002)

•When BH reaches AdS size, $E \sim M_{\rm Pl}^8 R^7 \Rightarrow$ at $E_R = M_{\rm Pl}(M_{\rm Pl} R)^7$.

•At $E \sim E_F \leftrightarrow E = \hat{E}_F$, depending on details of IR of gravity dual, BH reaches cut-off: max. behaviour: Froissart. (K. Kang, H.N., 2004)

•Simple argument (Giddings) linearized perturbation *on IR brane*.

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$$h_{00,lin} \sim G_4\sqrt{s}\frac{e^{-M_1 r}}{r} \sim 1; \quad G_4^{-1} = M_P^3 R$$
$$\Rightarrow r = r_H \Rightarrow \sigma \sim \pi r_H^2 \sim \frac{\pi}{M_1^2} ln^2(\sqrt{s}G_4 M_1)$$

•Here $M_1 = j_{1,1}/R =$ lightest KK mode on IR brane $\leftrightarrow$ glueball. For pion ($\sim$ fluctuation of D-brane position L)

$$\frac{\delta L}{L}|_{lin} \sim G_4\sqrt{s}(M_L R)\frac{e^{-M_1 r}}{r} \sim 1 \Rightarrow \sigma_{\rm QCD} \sim \frac{\pi}{M_L^2} ln^2(\sqrt{s}G_4 M_L R M_1)$$

•But, more precise: use *shockwaves* on IR brane. As $r \to \infty$, at $y = 0$ (IR brane), profile is

$$\Phi(r, y = 0) \simeq R_s\sqrt{\frac{2\pi R}{r}}C_1 e^{-M_1 r};$$
$$C_1 = \frac{j_{1,1}^{-1/2} J_2(j_{1,1})}{a_{1,1}}; \quad J_1(z) \sim a_{1,1}(z - j_{1,1}); \quad z \to j_{1,1}$$

•Exact condition for BH formation is the same (up to number of order 1) as the naive one, $\phi(b_{\text{max}}) \sim 1$, $\Rightarrow \sigma = \pi b_{\text{max}}^2$, namely

$$\sigma_{\text{tot}}(s) = \bar{K} \frac{\pi}{M_1^2} \ln^2[\sqrt{s} G_4 M_1 K], \quad K, \bar{K} \sim \mathcal{O}(1).$$

•More precisely, corrections: $\phi \sim 1 \Rightarrow$

$$\sigma_{\text{tot}}(s) = \frac{\pi}{M_1^2} \left[\ln(\sqrt{s} G_4 M_1 K) - \frac{1}{2} \ln \left(\ln(\sqrt{s} G_4 M_1 K) \right) \right]^2$$

•Exact map: $\phi(r, y = 0)$ to Heisenberg's DBI scalar. M_1 (first KK mode) $\leftrightarrow$ M of first glueball. Same $\phi \sim e^{-Mr}/\sqrt{Mr}$ behaviour.

•BH forming in collision of 2 gravitational shockwaves $\leftrightarrow$ pion field "soliton" (fireball) formed in collision of 2 pion field shockwaves.

•Experiments: "Soft Pomeron" (until 2001, in PDG) $\rightarrow$ small power law, $\sigma_{AB} \sim X_{AB} s^{\epsilon}$, $\sqrt{s} > 9GeV$. Later, used $Z_{AB} + B \log^2(s/s_0)$, extended down to $\sqrt{s} = 5GeV$, but with only *3 percent* better fit (χ^2/d.o.f) $\rightarrow$ simply $As^{\epsilon} \simeq A + \epsilon \log s + \epsilon^2/2 \log^2 s$.

•Find $B \sim 0.3mb$ instead of $\pi/m_\pi^2 \simeq 60mb$, but still $Z_{AB} \sim 18 - 65mb$ (now:*coincidence*!)

•Estimate E_F: is about a few TeV, such that

$$(s/s_0)^{\epsilon} = (\text{a few } TeV/9GeV)^{0.093} \sim 60mb/X_{AB} = \frac{\pi/m_\pi^2}{X_{AB}}$$

4. Uniqueness and generalizations

•Other higher derivative actions? For

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi)^2 + \sum_{n\geq 2} C_n[(\partial_\mu\phi)^2]^n ,$$

.

$\phi = A\sqrt{s}$ is not a solution:

$$\frac{A}{\sqrt{s}}(1+...) + \left(\frac{A}{2\sqrt{s}} - \frac{A}{2\sqrt{s}}\right)\sum_{n\geq 2}(...) = 0.$$

•Only for DBI coefficients is a solution:

$$\frac{A}{\sqrt{s}} + \frac{\#}{1 - A^4 l^2}\left(\frac{A}{2\sqrt{s}} - \frac{A}{2\sqrt{s}}\right) = 0$$

is rewritten as an identity.

•Can add $V(\phi)$ instead of $m^2\phi^2/2$ without changing the solution as $s \to 0$.

•Can consider N scalar mesons ϕ^i with σ model in $d+1$ dimensions, so DBI

$$\mathcal{L} = l^{-(d+1)}\left[h(\phi^i) - f(\phi)\sqrt{1 + l^{d+1}[g_{ij}(\phi^k)(\partial_\mu\phi^i)(\partial_\mu\phi^j) + 2V(\phi^i)]}\right]$$

•FInd $\phi^a = A^a\sqrt{s}$, $\sum_a (A^a)^2 = l^{-4}$, and

.

$$\sigma_a = \frac{\pi}{m_a^2} \ln^2 \left(\frac{\sqrt{s}}{\langle k_0 \rangle} \right)$$

•General D-brane action with $g_{ij}(\phi) \simeq g(\phi)\delta_{ij} \Rightarrow$ *on the solution* $\phi = \phi(s)$ we have the same action as before $\rightarrow$ same analysis.

•For AdS space ($m = 0$), $g(\phi) = \phi^{-4} \Rightarrow \phi(s) = 1/\sqrt{s}$ is exact solution. Also for general $g(\phi)$, $\int d\phi \sqrt{g(\phi)}$ is solution.

•Vector mesons:

$$\mathcal{L} = l^{-4}\left[1-\sqrt{\det(\eta_{ab}+l^4\partial_a\phi\partial_b\phi+l^2F_{ab})+m^2\phi^2+M_V^2A_a^2}\right]$$
$$= l^{-4}\left[1-\sqrt{1+l^4[(\partial_\mu\phi)^2+m^2\phi^2]+\frac{l^4}{2}F_{ab}F^{ab}-l^8\left(\frac{1}{4}\tilde{F}_{ab}F^{ab}\right)^2+M_V^2A_a^2+\ldots}\right]$$

•For just vector $A_a(r) = A_a e^{-M_V r}$. If also σ_V obtained in same way: $\langle k_0\rangle = \sqrt{s}e^{-M_V b_{\max}} \Rightarrow$

$$\sigma_V = \pi b_{\max}^2 = \frac{\pi}{M_V^2}\ln^2\frac{\sqrt{s}}{\langle k_0\rangle}$$

5. Pion wavefunction

•In general, $\phi = \phi(s, r)$, $r = \sqrt{y^2 + z^2}$ (transverse)

•Near $s = 0$, r finite, find $\phi = A\sqrt{s} + s^n f(r)$, $n \geq 1$.

•Near $s = 0$, $r = 0$, find $\phi = A\sqrt{s} + s^{3/2} f(r) + s^{5/2} g(r)$.

•Delta function shockwave? Find that $\phi = \delta(x^+)\phi(r)$ unphysical. Can add by hand *source* to action (but doesn't explain source-free shockwave).

•But: only need $T_{++} \to \infty$ at $x^+ = 0$. Usually: $\delta(x^+)$ in $T_{++} \Rightarrow \delta(x^+)$ in ϕ. Now: slow blow-up of $\mathcal{H}$ near $x^+ = 0$.

•Near $r \to \infty$, but a bit away, find *Bessel function* (solution to free wave eq.)

$$\phi = AK_0(mr) = A\frac{\pi i}{2}H_0^{(1)}(imr)$$

•At $r \to \infty$, find

$$\phi(r) \simeq A\sqrt{\frac{\pi}{2mr}}e^{-mr}$$

•At $r \to 0$, find

$$\phi(r) \simeq -A\ln\frac{mr}{2}$$

•Subleading corrections

$$\phi \simeq AK_0(mr) + \left(\frac{\pi}{2}\right)^{3/2}\frac{m^2l^4A^3}{4}\frac{e^{-3mr}}{(mr)^{3/2}}$$

6. Pion field source

•Source for pion shockwave: nucleons. How?

Vector BI

$$\mathcal{L} = \sqrt{1 + F - G^2} - 1$$
$$F \equiv \frac{1}{b^2}(\vec{B}^2 - \vec{E}^2); \qquad G \equiv \frac{1}{b^2}(\vec{B} \cdot \vec{E}).$$

•Define also

$$\vec{H} \equiv b^2 \frac{\partial \mathcal{L}}{\partial \vec{B}}; \qquad \vec{D} \equiv b^2 \frac{\partial \mathcal{L}}{\partial \vec{E}}$$

•Then find "Maxwell's equations"

$$\vec{\nabla} \times \vec{E} + \partial_0 \vec{B} = 0; \qquad \vec{\nabla} \cdot \vec{B} = 0$$
$$\vec{\nabla} \times \vec{H} - \partial_0 \vec{D} = 0; \qquad \vec{\nabla} \cdot \vec{D} = 0$$

•Then at $\vec{B} = \vec{H} = 0, \vec{\nabla} \cdot \vec{D} = 0$, which means

.
$$D_e = \frac{e}{r^2} \to \text{really} \quad \vec{\nabla} \cdot \vec{D} = 4\pi e \delta^3(r).$$

•$\vec{D}$ has *pointlike* source, though then

$$\rho = \frac{e}{2\pi r_0^3} \frac{1}{(r/r_0)(1 + (r/r_0)^4)^{3/2}}$$

is *spread out*.

•**Static scalar DBI**

$$\mathcal{L} = \sqrt{1 + \vec{F}^2}; \quad \vec{F} \equiv \vec{\nabla}\phi$$

•like the vector action for $\vec{B} = 0$, but different sign in r.

•Define $\vec{C} = \frac{\partial \mathcal{L}}{\partial \vec{F}}$, then $\vec{\nabla} \cdot \vec{C} = 0$, so find

.

$$C_r = \frac{e}{r^2} \to \text{really,} \quad \vec{\nabla} \cdot \vec{C} = 4\pi e \delta^3(r).$$

•**DBI scalar shockwave**

$$\mathcal{L} = l^{-4} \left[1 - \sqrt{1 - 4l^4 s \left(\frac{d\phi}{ds} \right)^2} \right]$$

•Define analog of electric field

$$E_s = 2\sqrt{s}\frac{d\phi}{ds}$$
$$\mathcal{L} = l^{-4} \left[1 - \sqrt{1 - E_s^2} \right]$$

•Define also $D_s = \partial\mathcal{L}/\partial E_s$, so equation of motion is

.
$$\frac{d}{ds}(\sqrt{s}D_s) = 0 \Rightarrow D_s = \frac{A}{\sqrt{s}}, \quad \sigma > 0: \quad D_s = \frac{A}{\sqrt{s}}\theta(s).$$

•At $s \to 0$, $\phi(s) \simeq l^{-2}\sqrt{s}\theta(s)$, and $E_s = l^{-2}\theta(s)$.

•But D charge is delta function, whereas E charge is spread out:

$$\begin{aligned} \frac{d}{ds}(\sqrt{s}D_s) &= A\delta(s) \\ \rho &= \frac{d}{ds}(\sqrt{s}E_s) = \frac{l^4A^2}{2\sqrt{s}(s + Al^4A^2)^{3/2}}\theta(s). \end{aligned}$$

7. The cross section

•Corrections to leading behaviour of $\sigma_{\text{tot}}(s) \to$ in parallel for Heisenberg model and gravitational dual (shockwave with profile ϕ)

$$\phi(r) \propto \frac{e^{-mr}}{\sqrt{mr}} \quad \text{vs.} \quad \Phi(r) \propto \frac{e^{-M_1 r}}{\sqrt{M_1 r}}$$

•One finds $\sigma_{\text{tot}}(s) = \pi b^2_{\text{max}}(s)$, with

$$b_{\text{max}} \simeq \frac{1}{m_\pi} \ln \frac{\sqrt{\tilde{s}}}{\langle k_{0,\pi} \rangle} - \frac{1}{2m_\pi} \ln \left[\ln \frac{\sqrt{\tilde{s}}}{\langle k_{0,\pi} \rangle} \right]$$

•New regime? If $l^{-1} \sim \Lambda_{\text{QCD}}$, $m = m_\pi$, and $(lm)^2 \ll 1$, then *linear* wavefunction, and never get into nonlinear regime: $\phi(r) \simeq -A \ln \frac{mr}{2}$, leading to ($\sqrt{\tilde{s}} > \langle k_{0,\pi} \rangle$)

$$\sigma_{\text{tot}}(\tilde{s}) = \pi b_{\text{max}}(\tilde{s})^2 = \frac{\pi}{m_\pi^2} e^{-4\frac{\langle k_{0,\pi} \rangle}{\sqrt{\tilde{s}}}}$$

•Black disk eikonal: $S = e^{i\delta}$, Im $\delta = \infty$ for $b \leq b_{\text{max}}(\tilde{s})$ and $\delta = 0$ for $b > b_{\text{max}}(\tilde{s})$. Then

$$\begin{aligned} \frac{1}{\tilde{s}}\mathcal{A}(\tilde{s},t) &= -i\int d^2b e^{i\vec{q}\cdot\vec{b}}\left(e^{i\delta(b,\tilde{s})}-1\right) \\ &= 2\pi i\frac{b_{\text{max}}(\tilde{s})}{\sqrt{t}}J_1(\sqrt{t}b_{\text{max}}(\tilde{s})) \end{aligned}$$

•Total cross section is

$$\frac{1}{\tilde{s}}\text{Im}\mathcal{A}_{\text{elastic}}(\tilde{s},t=0) = \sigma_{\text{tot}}(k_1,k_2 \to \text{anything}) = \pi b_{\text{max}}(\tilde{s})^2$$

•But differential cross section is

$$\frac{d\sigma}{dt} = \frac{|\mathcal{A}(\tilde{s},t)|^2}{16\pi\tilde{s}(\tilde{s}-4m^2)}.$$

•For black disk eikonal amplitude, find

$$\frac{\sigma_{\text{elastic}}}{\sigma_{\text{tot}}} = \frac{1}{4}\left[1-\left(J_0(|b_{\text{max}}(\tilde{s})|\sqrt{\tilde{s}-4m^2})\right)^2-\left(J_1(|b_{\text{max}}(\tilde{s})|\sqrt{\tilde{s}-4m^2})\right)^2\right].$$

$$\simeq \frac{1}{4}\left[1-\frac{2}{\pi b_{\text{max}}(\tilde{s})\sqrt{\tilde{s}-4m_N^2}}\right]$$

•Translating into QCD, with Heisenberg's model,

$$\frac{\sigma_{\text{elastic}}}{\sigma_{\text{tot}}} \simeq \frac{1}{4}\left[1-\frac{2m_\pi}{\pi \ln(\tilde{s}/s_0)\sqrt{\tilde{s}-4m_N^2}}\right]$$

•Compares well with experimental results.

8. Holographic models

•One model: hard-wall (Polchinski-Strassler)

•But any gravity dual can be approximated by it. Details of IR only matter for $E_F \rightarrow$ energy at which Froissart behaviour onsets. Otherwise general.

•Note that ∞ nr. of higher derivative terms in DBI $\rightarrow$ chiral perturbation theory $\leftrightarrow$ geometry of D-brane embedding

•Other model: Sakai-Sugimoto type.

•Consider a double Wick rotation of a nonextremal D4 background, with a $D8 - \bar{D}8$ pair forming a U-shape.

$$ds^2 = \left(\frac{u}{R_{D4}}\right)^{3/2}\left[-dt^2 + \delta_{ij}dx^i dx^j + f(u)dx_4^2\right] + \left(\frac{R_{D4}}{u}\right)^{3/2}\left[\frac{du^2}{f(u)} + u^2 d\Omega_4^2\right]$$

$$F_4 = \frac{2\pi N_c}{V_4}\epsilon_4 \quad , \quad e^{\phi} = g_s\left(\frac{u}{R_{D4}}\right)^{3/4}, \; R_{D4}^3 = \pi g_s N_c l_s^3 \; , \quad f(u) = 1 - \left(\frac{u_\Lambda}{u}\right)^3$$

•The D8-brane action, integrated over transverse coordinates, is

$$S_{DBI} = \tilde{T}_8 \int dt d^3x dx_4 \, \phi^4 \sqrt{f(\phi) + \left(\frac{R_{D4}}{\phi}\right)^3 \left[\partial_\mu\phi\partial^\mu\phi + \frac{1}{f(\phi)}(\partial_{x_4}\phi)^2\right]}$$

•Expand $\phi(x^\mu, x^4)$ in KK modes and integrate over massive modes (with wavefunction $\zeta_n(x_4)$) $\rightarrow$ leave an action for the lowest one, assumed to be of the type

$$S_{DBI} = \tilde{T}_8 \int dt d^3x \phi^4 \sqrt{f(\phi) + \left(\frac{R_{D4}}{\phi}\right)^3 \left[\partial_\mu\phi\partial^\mu\phi + m^2\phi^2\right]}.$$

•Protons $\rightarrow$ can be thought of as wrapped D4-branes on the flavor brane giving an instanton-like structure, and action like the one above.

9. Relativistic hydrodynamics and DBI

•We want to obtain a hydro description *from* the Heisenberg one via the shock of the shockwave solution.

•Belinfante energy-momentum tensor: $T_{00} \to \infty$ at shock.

•Expanding DBI, find coefficient of $\phi^2(\partial_\mu\phi)^2$ and identify with $f_\pi^{-2}/6$, giving

$$\beta \equiv l^2 = \frac{\sqrt{2}}{\sqrt{3}m_\pi f_\pi} \simeq \frac{1}{(126MeV)^2}$$

•Relativistic hydro: expansion in ∂u $(a_\mu = u^\rho \nabla_\rho u_\mu)$

$$\begin{aligned} T_{\mu\nu} &= \rho u_\mu u_\nu + P(g_{\mu\nu} + u_\mu u_\nu) + \pi_{\mu\nu} \\ \pi_{\mu\nu} &= -2\eta\left[\frac{\nabla_\mu u_\nu + \nabla_\nu u_\mu}{2} + \frac{a_\mu u_\nu + a_\nu u_\mu}{2} - \frac{1}{3}(\nabla^\rho u_\rho)(g_{\mu\nu} + u_\mu u_\nu)\right] \\ &\quad -\zeta(\nabla^\mu u_\mu)(g_{\mu\nu} + u_\mu u_\nu) + \ldots \end{aligned}$$

•Eqs. of motion (relativistic generalization of Navier-Stokes) are $\partial^\mu T_{\mu\nu} = 0$.

•Obtain (in non-rel. limit: Navier-Stokes)

$$\begin{aligned} & u_\mu u_\nu \partial^\mu(\rho + P) + (\rho + P)(\partial^\mu u_\mu)u_\nu + (\rho + P)u_\mu \partial^\mu u_\nu + \partial_\nu P \\ = \; & \eta \partial^\mu(\partial_\mu u_\nu + \partial_\nu u_\mu) + \left(\zeta - \frac{2\eta}{3}\right)\partial^\mu\left[(\partial^\rho u_\rho)(g_{\mu\nu} + u_\mu u_\nu)\right] \end{aligned}$$

•Conformal fluids: If $T^\mu{}_\mu = 0 \Rightarrow \rho = 3P, \zeta = 0$.

•But for $d \neq 2$, conf. inv. $\not\Rightarrow T^\mu{}_\mu = 0$. E.g. massless scalar:

$$S = \int d^4x \left[-\frac{1}{2}(\partial_\mu \phi)^2\right] \Rightarrow T^{B\mu}{}_\mu = \frac{2-d}{2}(\partial_\rho \phi)^2.$$

•Use Noether ambiguity $T^{\mu\nu} \to T^{\mu\nu} + \partial_\lambda J^{\mu\nu\lambda}$.

•Now

$$T^{I}_{\mu\nu} = T^{B}_{\mu\nu} - \frac{1}{6}(\partial_\mu\partial_\nu - g_{\mu\nu}\partial^2)\phi^2$$

is traceless on-shell ($\Box\phi = 0$)

$$T^{I\mu}{}_{\mu} = T^{B\mu}{}_{\mu} + (\partial_\mu\phi)^2 + \phi\partial^2\phi$$

10. DBI scalar hydrodynamics

•Heisenberg model worked well in sQGP regime ($s \to \infty$), so hydro description should *follow* from it.

•What is u^μ? $u^\mu = k^\mu/m \propto \partial^\mu \phi$:

$$u^\mu = \frac{\partial^\mu \phi}{\sqrt{-(\partial_\mu \phi)^2}}$$

•Consistent with usual scalar field $\int d^2x[-(\partial\phi)^2/2 - V]$. Then $u^\mu = (1,0,0,0)$ and

$$\rho_\phi = \frac{\dot{\phi}^2}{2} + V; \quad p_\phi = \frac{\dot{\phi}^2}{2} - V$$

•Free massless scalar: use improved $T_{\mu\nu}$. Then, identify with $T_{\mu\nu}$ expansion in $u^\mu \Rightarrow$

$$\rho = -\frac{1}{2}(\partial_\mu \phi)^2; \quad P = -\frac{1}{6}(\partial_\mu \phi)^2 \Rightarrow P = \rho/3$$

•Identifying the coefficient of the $\partial_\mu u_\nu$ term, for the free massless scalar, we find

$$\zeta = 0; \quad 2\eta = \frac{\phi}{3}\sqrt{-(\partial_\lambda \phi)^2} \to \infty \text{ as } s \to 0.$$

•However, the important object is the ratio

$$\frac{\eta}{s} = \frac{\eta T}{\rho + P} = \frac{\phi T}{4\sqrt{-(\partial_\lambda \phi)^2}} \to 0; \text{ as } s \to 0.$$

•Non-viscous fluid.

•Ideal DBI hydro: generalize to DBI.

•Improved tensor that is traceless as $\beta \to 0$ is

$$T^I_{\mu\nu} = \frac{\frac{2}{3}(\partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}(\partial_\rho\phi)^2) - g_{\mu\nu}\beta^{-2}}{\sqrt{1+\beta^2(\partial_\lambda\phi)^2}} + \frac{g_{\mu\nu}}{\beta^2}$$
$$-\frac{\phi}{3\sqrt{1+\beta^2(\partial_\lambda\phi)^2}}\left(\partial_\mu\partial_\nu\phi - \frac{\beta^2\partial_\nu\phi(\partial_\mu\partial^\rho\phi)\partial_\rho\phi}{1+\beta^2(\partial_\lambda\phi)^2}\right).$$

•The velocity is the same $u^\mu = \partial^\mu\phi/\sqrt{-(\partial_\lambda\phi)^2} \to \beta^2\partial^\mu\phi$ as $s \to 0$.

•Find

$$\rho = \frac{1}{\beta^2\sqrt{1+\beta^2(\partial_\lambda\phi)^2}} - \frac{1}{\beta^2}$$
$$P = \frac{1}{\beta^2}(1-\sqrt{1+\beta^2(\partial_\lambda\phi)^2}) + \frac{(\partial_\rho\phi)^2}{3\sqrt{1+\beta^2(\partial_\lambda\phi)^2}}$$

•But that gives $(3P+\rho)/\rho \to 0$ ($P \simeq -\rho/3$). Wrong.

•With another improved tensor, we get tracelessness near the shock:

$$
\begin{aligned}
T^{I}_{\mu\nu} &= \frac{\frac{4}{3}(\partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}(\partial_\rho\phi)^2) - g_{\mu\nu}\beta^{-2}}{\sqrt{1+\beta^2(\partial_\lambda\phi)^2}} + \frac{g_{\mu\nu}}{\beta^2} \\
&+ \frac{\phi}{3\sqrt{1+\beta^2(\partial_\lambda\phi)^2}}\left(\partial_\mu\partial_\nu\phi - \frac{\beta^2\partial_\nu\phi(\partial_\mu\partial^\rho\phi)\partial_\rho\phi}{1+\beta^2(\partial_\lambda\phi)^2}\right).
\end{aligned}
$$

•Find

$$
\begin{aligned}
\rho &= \frac{1}{\beta^2\sqrt{1+\beta^2(\partial_\lambda\phi)^2}} - \frac{1}{\beta^2} \\
P &= \frac{1}{\beta^2}(1-\sqrt{1+\beta^2(\partial_\lambda\phi)^2}) - \frac{(\partial_\rho\phi)^2}{3\sqrt{1+\beta^2(\partial_\lambda\phi)^2}} \simeq \frac{\rho}{3} \to \infty
\end{aligned}
$$

•Then we find viscosity:

$$\eta = \frac{\phi\sqrt{-(\partial_\lambda \phi)^2}}{3\sqrt{1+\beta^2(\partial_\mu \phi)^2}}$$

•But near the shock

$$\frac{\eta}{\rho + P} = \frac{\phi}{4\sqrt{-(\partial_\lambda \phi)^2}} \to 0$$

•So DBI is "ideal hydro"

•Can add mass m and conclusions don't change.

•Viscous DBI: add term inside the square root of DBI

•Modified DBI is

$$S = -\beta^{-2}\int d^4x\left[\sqrt{1+\beta^2[(\partial\phi)^2+m^2\phi^2]+\alpha\left[\partial^2\phi-\frac{(\partial_\mu\phi)(\partial^\mu\partial^\rho\phi)(\partial_\rho\phi)}{(\partial_\lambda\phi)^2}\right]}-1\right]$$

•On the $\phi=\phi(s)$ solution, it becomes

$$S = -\beta^{-2}\int d^4x\left[\sqrt{1+\beta^2\left(-4s\left(\frac{d\phi}{ds}\right)^2+m^2\phi^2\right)+\alpha\left(-2\frac{d\phi}{ds}\right)}-1\right]$$

•Shear viscosity is then

$$2\eta = \frac{\alpha\sqrt{-(\partial_\lambda\phi)^2}}{\beta^2\sqrt{1+\beta^2[(\partial\phi)^2+m^2\phi^2]+\alpha\left[\partial^2\phi-\frac{(\partial_\mu\phi)(\partial^\mu\partial^\rho\phi)(\partial_\rho\phi)}{(\partial_\lambda\phi)^2}\right]}}\to\infty$$

and bulks is $\zeta = 2\eta/3$.

•Obs. Relativistic Navier-Stokes is *satisfied*.

$$\frac{\eta}{s} = \frac{\eta T}{\rho + P} = T\frac{\alpha}{\beta^2\sqrt{-(\partial_\lambda\phi)^2}} \to T\frac{\alpha}{\beta}$$

is finite!

•Fix parameters from experiment. On-shell,

$$\sim \frac{\alpha^2}{\beta^2}(\partial^2\phi)^2 \sim \frac{\alpha^2 m_\pi^4}{\beta^2}\phi^2 \leftrightarrow \sim m^2\phi^2.$$

giving

$$\alpha \sim \frac{\beta}{m_\pi} = \frac{\sqrt{2}}{\sqrt{3}f_\pi m_\pi^2}.$$

•Then also

$$T \sim m_\pi\frac{\eta}{s} \sim m_\pi\left(\times\frac{1}{4\pi}\right)$$

•If $T \simeq 4m_\pi/\pi \Rightarrow$ conjectured in HN 2015, then

$$\alpha \simeq \frac{\beta}{16 m_\pi}$$

•Viscosity regulates shock. Solution $\phi \simeq \sqrt{s}/\beta$ not valid anymore. We have instead

$$\phi(s) \simeq A + BsCs^2 + ..., \quad \sqrt{s} \ll \frac{3\alpha}{4\beta} \propto \frac{1}{m_\pi}$$

and the solution $\phi(s) \simeq \sqrt{s}/\beta$ valid for (if $\exists$ small nrs. like 1/16 above)

$$\frac{3\alpha}{4\beta} \ll \sqrt{s} \ll \frac{1}{m_\pi}$$

•Extra term regulates behaviour: denominator is finite in $\mathcal{H} \to$ finite energy density at shock.

Conclusions

•Heisenberg model describes well $s \to \infty$ regime: Froissart bound like in AdS/CFT.

•Is unique, but can be generalized by including $V(\phi)$, several scalar mesons, vector mesons, extra functions.

•For near-asymptotic wavefunction, obtain corrections to $\sigma_{\text{tot}}(s)$ away from Froissart.

•The shockwave has dual interpretation: delta function at shock $x^+ = 0$, and spread out, similar to vector DBI.

•We can also find $\sigma_{\text{elastic}}/\sigma_{\text{tot}}(s)$ under certain assumptions

•Ideal hydrodynamics is derived from DBI.

•Can introduce finite η as a regulator extra term in DBI, and then $\zeta = 2\eta/3$.