**GUASA 2015: Cosmology**
**Density perturbations in an expanding universe;**
**the correlation function**

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## 1 Gravitational instability in an expanding universe

As we have seen, density perturbations smaller than the Hubble distance can grow only if they are not pressure-supported. For the baryonic matter, the loss of pressure support happens abruptly at the time of decoupling, when the Jeans length drops by a factor of $\sim 10^{-5}$. Recall that the Jeans length is set by the ability of pressure support traveling at the sound speed to counteract gravitational collapse. For a density perturbation to be stabilized by pressure against collapse, it must be smaller than a size given by the relation

$$\lambda_J \sim c_s t_{\rm dyn} \sim c_s \left( \frac{c^2}{G\bar{u}} \right)^{1/2} . \tag{1}$$

Overdense regions larger than the Jeans length collapse under their own gravity, while overdense regions smaller than the Jeans length oscillate in density as stable sound waves.

For the dark matter, the loss of pressure support is more gradual, as the thermal energy of the dark matter particles drops below their rest energy. If the dark matter (WIMPs, presumably) has mass $m_W c^2 \gg 2$ eV, it would have become nonrelativistic well before the time of matter-radiation equality at $z \sim 3300$.

Once the pressure and hence the Jeans length of some component of the universe becomes negligibly small, it isn't necessarily the case that density fluctuations in that component can grow exponentially with time, however. This is because our earlier analysis assumed that the universe was static as well as pressureless. In the real universe, we need to consider the effects of expansion.

In an expanding universe described by the Friedmann equation, the timescale for the growth of a density perturbation by self-gravity,

$$t_{\rm dyn} \sim \left( \frac{1}{G\bar{u}} \right)^{1/2} , \tag{2}$$

is comparable to the timescale for expansion,

$$H^{-1} \sim \left( \frac{1}{G\bar{u}} \right)^{1/2} \tag{3}$$

(we have left out the numerical factors in both expressions).

Self-gravity causes overdense regions to become more dense with time, while the global expansion of the universe causes them to become less dense. Because the timescales for these processes are similar, we need to take both of them into account when computing the time evolution of a density perturbation.

We will do a Newtonian analysis of this problem. Suppose we are in a universe filled with pressureless matter with mass density $\bar{\rho}$. As the universe expands, the density decreases as $\bar{\rho}(t) \propto a(t)^3$. Within a spherical region of radius $R$ a small amount of matter is added or removed, so that the density in the sphere is

$$\rho(t) = \bar{\rho}[1 + \delta(t)], \tag{4}$$

with $|\delta| \ll 1$. We are assuming that the radius $R$ is small compared to the Hubble volume and large compared to the Jeans length.

The total gravitational acceleration at the surface of the sphere will be

$$\ddot{R} = -\frac{GM}{R^2} = -\frac{G}{R^2}\left(\frac{4\pi}{3}\rho R^3\right) = -\frac{4\pi}{3}G\bar{\rho}R - \frac{4\pi}{3}G(\bar{\rho}\delta)R. \tag{5}$$

We can then write the equation of motion for a point on the surface of the sphere as

$$\frac{\ddot{R}}{R} = -\frac{4\pi}{3}G\bar{\rho} - \frac{4\pi}{3}G\bar{\rho}\delta. \tag{6}$$

The mass inside the sphere is

$$M = \frac{4\pi}{3}\bar{\rho}[1+\delta(t)]R(t)^3, \tag{7}$$

and it must remain constant as the sphere expands. So

$$R(t) \propto \bar{\rho}(t)^{-1/3}[1+\delta(t)]^{-1/3}, \tag{8}$$

or, since $\bar{\rho} \propto a^{-3}$,

$$R(t) \propto a(t)[1+\delta(t)]^{-1/3}. \tag{9}$$

This tells us that if the sphere is slightly overdense, its radius will grow slightly less rapidly than the scale factor, while if it is under dense it will grow slightly more rapidly than the scale factor.

Taking two time derivatives of Equation 9, we get

$$\frac{\ddot{R}}{R} = \frac{\ddot{a}}{a} - \frac{1}{3}\ddot{\delta} - \frac{2}{3}\frac{\dot{a}}{a}\dot{\delta}, \tag{10}$$

when $|\delta| \ll 1$. Combining this with Equation 6, we find

$$\frac{\ddot{a}}{a} - \frac{1}{3}\ddot{\delta} - \frac{2}{3}\frac{\dot{a}}{a}\dot{\delta} = -\frac{4\pi}{3}G\bar{\rho} - \frac{4\pi}{3}G\bar{\rho}\delta. \tag{11}$$

If $\delta = 0$, this reduces to

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}G\bar{\rho}, \tag{12}$$

which is the acceleration equation for a homogeneous and isotropic universe containing only pressureless matter.

We then subtract Equation 12 from Equation 11 to leave only the terms involving $\delta$. This gives us the equation describing the growth of small density perturbations:

$$-\frac{1}{3}\ddot{\delta} - \frac{2}{3}\frac{\dot{a}}{a}\dot{\delta} = -\frac{4\pi}{3}G\bar{\rho}\delta, \tag{13}$$

which is

$$\ddot{\delta} + 2H\dot{\delta} = 4\pi G\bar{\rho}\delta. \tag{14}$$

In a static universe with $H = 0$, this reduces to

$$\ddot{\delta} = 4\pi G\bar{\rho}\delta \tag{15}$$

which we derived in Lecture 16 (Equation 11) under the assumption of a static universe. The additional term proportional to $H\dot{\delta}$ is sometimes called the "Hubble friction" term, since it slows the growth of density perturbations in an expanding universe.

A fully relativistic calculation of the growth of density perturbations gives the similar expression

$$\ddot{\delta} + 2H\dot{\delta} = \frac{4\pi G}{c^2}\bar{u}_m \delta. \tag{16}$$

This can be applied to universes that include components with non-negligible pressure, such as radiation or a cosmological constant. It is important to remember that in such universes $\delta$ represents the fluctuation in the density of matter only, however:

$$\delta = \frac{u_m - \bar{u}_m}{\bar{u}_m}, \tag{17}$$

where the average energy density of matter $\bar{u}_m(t)$ might be only a small part of the total average density $\bar{u}(t)$.

Rewritten in terms of the matter density parameter

$$\Omega_m = \frac{\bar{u}_m}{u_c} = \frac{8\pi G \bar{u}_m}{3c^2 H^2}, \tag{18}$$

Equation 16 is

$$\ddot{\delta} + 2H\dot{\delta} - \frac{3}{2}\Omega_m H^2 \delta = 0. \tag{19}$$

During epochs when the universe isn't matter-dominated, density perturbations in the matter don't grow rapidly in amplitude. For example, in the radiation-dominated phase, $\Omega_m \ll 1$ and $H = 1/(2t)$, so Equation 19 is

$$\ddot{\delta} + \frac{1}{t}\dot{\delta} \approx 0, \tag{20}$$

which has the solution

$$\delta(t) \approx B_1 + B_2 \ln t. \tag{21}$$

So in the radiation-dominated era, density fluctuations grew only at a logarithmic rate.

We can also consider the far future. If the universe is dominated by a cosmological constant, the matter density parameter will become negligibly small, the Hubble parameter will have the constant value $H = H_\Lambda$, and Equation 19 will have the form

$$\ddot{\delta} + 2H_\Lambda \dot{\delta} \approx 0. \tag{22}$$

The solution to this is of the form

$$\delta(t) \approx C_1 + C_2 e^{-2H_\Lambda t}. \tag{23}$$

In an epoch of the universe dominated by the cosmological constant, fluctuations in the matter density approach a constant fractional amplitude, while the average matter density decreases exponentially.

Fluctuations in the matter density can grow significantly only when the universe is matter-dominated. If the universe is flat with $\Omega_m = 1$, then $H = 2/(3t)$ and Equation 19 is

$$\ddot{\delta} + \frac{4}{3t}\dot{\delta} - \frac{2}{3t^2}\delta = 0. \tag{24}$$

We will guess that the solution to this equation has the power law form $Dt^n$. Putting this guess into the equation, we have

$$n(n-1)Dt^{n-2} + \frac{4}{3t}nDt^{n-1} - \frac{2}{3t^2}Dt^n = 0, \tag{25}$$

which is

$$n(n-1) + \frac{4}{3}n - \frac{2}{3} = 0. \tag{26}$$

The two possible solutions to this equation are $n = -1$ and $n = 2/3$. Therefore the general solution for the time evolution of density perturbations in a flat, matter-only universe is

$$\delta(t) \approx D_1 t^{2/3} + D_2 t^{-1}. \tag{27}$$

The values of $D_1$ and $D_2$ are determined by the initial conditions for $\delta(t)$.

The decaying mode proportional to $t^{-1}$ will eventually become negligible compared to the growing mode proportional to $t^{2/3}$. At this point, the density perturbations in a flat, matter-only universe will grow at the rate

$$\delta \propto t^{2/3} \propto a(t) \propto \frac{1}{1+z} \tag{28}$$

as long as $|\delta| \ll 1$.

For different cosmological parameters, it can be shown that density perturbations grow according to

$$D_+(a) \propto \frac{H(a)}{H_0} \int_0^a \frac{da}{[\Omega_m/a + \Omega_\Lambda/a^2 - (\Omega_m + \Omega_\Lambda - 1)]^{3/2}}, \tag{29}$$

where $D_+(a)$ is the growing solution to Equation 19. For non-Einstein-de Sitter cosmologies, we do not find $\delta(t) \propto a(t)$, but the qualitative behavior is similar. In particular, fluctuations were able to grow by a factor $\sim 1000$ from the epoch of recombination at $z \sim 1000$ to today.

At the present time $\delta \gg 1$ on scales of clusters of galaxies ($\sim 2$ Mpc), and $\delta \sim 1$ on scales of superclusters ($\sim 10$ Mpc). Since the density fluctuations have grown by a factor of $\sim 1000$ since $z \sim 1000$, we would expect $\delta \gtrsim 10^{-3}$ at recombination in order to have non-linear ($\delta \gg 1$) structures today. This leads us to expect that the anisotropies in the CMB should also be of order $\Delta T/T \gtrsim 10^{-3}$. However, the observed amplitude of the fluctuations is $\Delta T/T \sim 10^{-5}$; density fluctuations of this magnitude cannot have grown enough to produce the structures we see today.

The solution to this problem is the dominance of dark matter, which allows the process of structure formation to begin earlier. The CMB anisotropies (at least on scales of less than $\sim 1°$), provide information on the density contrast of *baryons*. Perturbations in the baryonic matter couldn't begin to grow until after decoupling, while the dark matter perturbations could grow as soon as the universe became matter-dominated. At decoupling, the baryons fell into the already existing gravitational wells of dark matter. This resolves at least part of the problem of how structure formed so early in the history of the universe.

When an overdensity reaches $\delta \sim 1$, the linear analysis we have carried out above no longer applies. At that point the overdensity breaks free from the Hubble flow and collapses, eventually reaching virial equilibrium as a gravitationally bound structure. If the baryons in the structure are able to cool efficiently, they will radiate energy and fall to the center where they eventually form stars, becoming the visible parts of the galaxies we see today. The less concentrated dark matter forms the halo in which the stellar component of the galaxy is embedded. The problem of structure formation is generally treated with large numerical simulations, in which the matter in the universe is modeled as a distribution of point sources interacting via Newtonian gravity. We will discuss these simulations in more detail later.

## 2 The correlation function

As with the cosmic microwave background, we are interested not in the under- or overdensity of a particular point in the universe, but in the statistical properties of the density fluctuations on different scales. We will discuss two related statistical descriptions of the density field.

Redshift surveys show that galaxies are not randomly distributed in space; instead they are clustered in groups, clusters and superclusters. This means that the probability of finding a galaxy at location $x$ is not independent of whether or not there is a galaxy in the vicinity of $x$. We are more likely to find a galaxy near another galaxy than at an arbitrary location.

We describe this by considering two points $\boldsymbol{x}$ and $\boldsymbol{y}$, and two volume elements $dV$ around these points. If $\bar{n}$ is the average number density of galaxies, the probability of finding a galaxy in the volume element $dV$ around $\boldsymbol{x}$ is

$$P_1 = \bar{n}\, dV, \tag{30}$$

independent of $\boldsymbol{x}$ if we assume that the universe is statistically homogenous. We choose $dV$ such that $P_1 \ll 1$, so that the probability of finding two or more galaxies in the volume element $dV$ is negligible.

The probability of finding a galaxy in the volume element $dV$ at location $\boldsymbol{x}$ and simultaneously finding a galaxy in the volume element $dV$ at location $\boldsymbol{y}$ is then

$$P_2 = (\bar{n}\, dV)^2 [1 + \xi_g(\boldsymbol{x}, \boldsymbol{y})]. \tag{31}$$

If the distribution of galaxies was uncorrelated, the probability $P_2$ would just be the product of the probabilities of finding a galaxy at each of the locations $\boldsymbol{x}$ and $\boldsymbol{y}$ in a volume element $dV$, so $P_2 = P_1^2$. Since the distribution is correlated, this does not apply, and we modify the probability as shown in Equation 31. The quantity $\xi_g(\boldsymbol{x}, \boldsymbol{y})$ is the **two-point correlation function**.

Similarly, we can define the correlation function for the total matter density:

$$\begin{aligned} \langle \rho(\boldsymbol{x}) \rho(\boldsymbol{y}) \rangle &= \bar{\rho}^2 \langle [1 + \delta(\boldsymbol{x})][1 + \delta(\boldsymbol{y})] \\ &= \bar{\rho}^2 (1 + \langle \delta(\boldsymbol{x}) \delta(\boldsymbol{y}) \rangle) \\ &= \bar{\rho}^2 [1 + \xi(\boldsymbol{x}, \boldsymbol{y})], \end{aligned} \tag{32}$$

since $\langle \delta(\boldsymbol{x}) \rangle = 0$ for all locations $\boldsymbol{x}$.

Since the universe is homogeneous, the correlation function $\xi$ can depend only on the difference $\boldsymbol{x} - \boldsymbol{y}$ and not on $\boldsymbol{x}$ and $\boldsymbol{y}$ individually. $\xi$ can also depend only on the separation $r = |\boldsymbol{x} - \boldsymbol{y}|$ and not on the direction of the separation vector $\boldsymbol{x} - \boldsymbol{y}$, since we assume the universe is statistically isotropic. Therefore $\xi = \xi(r)$ is simply a function of the separation between two points, and the correlation function is a measure of the excess probability (relative to a Poisson distribution) of finding an object in a volume element $dV$ at a separation $r$ from another randomly chosen object.

This can be seen when the correlation function is written

$$\xi(r) = \frac{DD(r)\Delta r}{RR(r)\Delta r} - 1, \tag{33}$$

where $DD(r)\Delta r$ is the number of galaxy pairs with separations in the range $r \pm \Delta r/2$ and $RR(r)\Delta r$ is the number that would be expected if galaxies were randomly distributed in space. Galaxies are said to be positively correlated on a scale $r$ if $\xi(r) > 0$, to be anticorrelated if $\xi(r) < 0$, and to be uncorrelated if $\xi(r) = 0$.

The correlation function can be measured from spectroscopic surveys of the redshifts of galaxies. It is found to be well described by a power law

$$\xi_g = \left( \frac{r}{r_0} \right)^{-\gamma}, \tag{34}$$

where $r_0$ is the *correlation length*. Galaxies with separations larger than the correlation length are relatively uncorrelated.

The correlation length $r_0$ and slope $\gamma$ may vary for different populations of galaxies; in particular, brighter and redder galaxies are found to be more strongly clustered than bluer and fainter galaxies. An example of the correlation function is shown in Figure 1. The best-fitting correlation function for this sample (luminous red galaxies from the Sloan Digital Sky Survey) gives $r_0 = 5.59h^{-1}$ Mpc and $\gamma = 1.84$. The exponent $\gamma$ is related to the initial spectrum of density fluctuations.

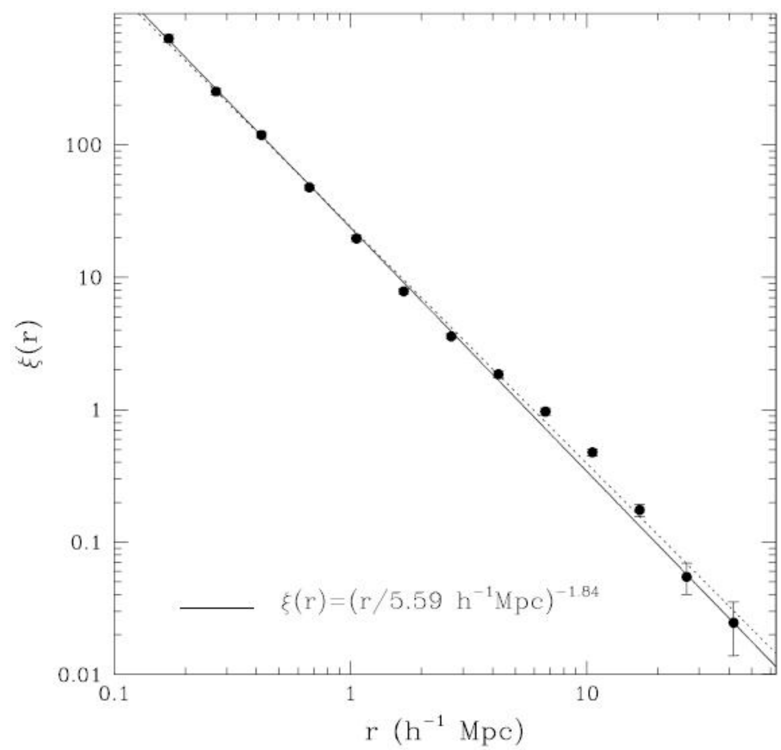


Figure 1: The correlation function for luminous red galaxies from the Sloan Digital Sky survey.