# GUASA 2015: Cosmology
# CMB anisotropies

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The familiar CMB power spectrum is shown in Figure 1. Today we will discuss the origins of the peaks and troughs, and what we can learn from them.

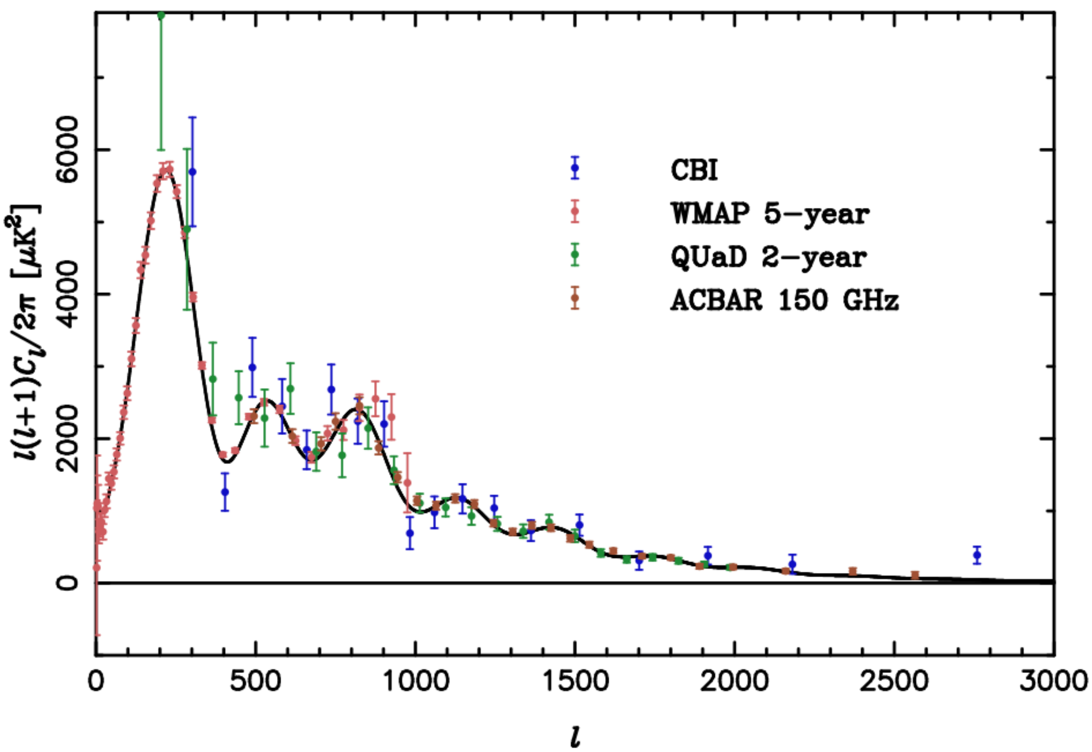


Figure 1: The CMB power spectrum, including data from four recent experiments. From Sievers et al. (2009).

## 1 Description of temperature fluctuations: review

First a brief review of how we describe the fluctuations:

- We are interested in the statistical properties of the temperature fluctuations; the strength of fluctuations on different angular scales over different parts of the sky. Since the temperature fluctuations $\delta T/T$ are observed on the spherical surface of last scattering, it is useful to expand in spherical harmonics:

$$\frac{\delta T}{T}(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\theta, \phi) \tag{1}$$

  where $Y_{lm}$ are the usual spherical harmonic functions.

- Most important statistical property is the correlation function $C(\theta)$. We consider two points in directions $\hat{n}$ and $\hat{n}'$, separated by an angle $\theta$ given by $\cos\theta = \hat{n}\cdot\hat{n}'$. Then the correlation function is

$$C(\theta) = \left\langle \frac{\delta T}{T}(\hat{n})\frac{\delta T}{T}(\hat{n}') \right\rangle_{\hat{n}\cdot\hat{n}'=\cos\theta} \tag{2}$$

  We multiply $\delta T/T$ at the two points and average the product over all points separated by the angle $\theta$.

- We would like to know the value of $C(\theta)$ over all angles from $\theta = 0$ to $\theta = 180^\circ$, but we are limited by the range of angular scales we can actually measure.

- Limited range of angular resolution available makes the expansion in spherical harmonics very useful. We can write the correlation function in the form

$$C(\theta) = \frac{1}{4\pi}\sum_{l=0}^{\infty}(2l+1)C_l P_l(\cos\theta) \tag{3}$$

  where $P_l$ are the Legendre polynomials. This allows us to break a measured correlation function $C(\theta)$ into its multipole moments $C_l$.

- For a given experiment, $C_l$ will be nonzero for angular scales larger than the resolution of the experiment and smaller than the patch of sky being examined. In general, a term $C_l$ is a measurement of temperature fluctuations on the angular scale $\theta \sim 180^\circ/l$, so for practical purposes the multipole $l$ is interchangeable with the angular scale $\theta$.

- The $l = 0$ (monopole) term vanishes if we have the right mean temperature. The $l = 1$ (dipole) term primarily results from our motion through space as already discussed. Moments with $l \geq 2$ are most interesting for cosmology, since they tell us about fluctuations at time of last scattering.

- In presenting results of CMB observations, customary to plot the function

$$\Delta_T \equiv \left(\frac{l(l+1)}{2\pi}C_l\right)^{1/2}\langle T\rangle. \tag{4}$$

  This tells us the contribution per logarithmic interval in $l$ to the total fluctuations $\delta T$ of the CMB. This tells us where the energy is across a wide range of $l$.

## 2 Origin of the fluctuations

Now we'll discuss the origin of the fluctuations and how they relate to the shape of the CMB power spectrum.

### 2.1 The universe before recombination

- Because the positions of particles are indeterminate, the universe is always filled with density fluctuations: regions which are randomly underdense or overdense relative to the average density.

- Inflation took density variations on very small scales and expanded them to super-horizon size. Fluctuations on super-horizon scales are no longer in causal contact, and so cannot respond to changes in the environment; they are *frozen*.

- When particle horizon grows large enough to encompass the density fluctuations, they can then react to the environment.

- The size of a region that can respond as a whole is determined by the time available for a sound wave to cross the region. This is called the **sonic horizon** or **acoustic horizon**.

- The sound speed is
$$c_s = \left(\frac{\partial P}{\partial \rho}\right)^{1/2}, \tag{5}$$
so it depends on the equation of state $P = w\rho c^2$. Before decoupling, photons, electrons and baryons were tightly coupled in a **photon-baryon** fluid, and the sound speed is
$$c_s = c/\sqrt{3} \tag{6}$$
as expected for a photon gas with $w = 1/3$.

- Therefore the sonic horizon $d_s$ is related to the particle horizon $d_h$ by
$$d_s = d_h/\sqrt{3} \tag{7}$$

- Decoupling occurred in the matter-dominated era when the particle horizon is $d_h = 3ct$. Recall: the horizon distance at time $t$ is
$$d_h(t) = a(t)\int_0^t \frac{c\,dt}{a(t)}. \tag{8}$$
During the matter era (assuming a flat universe with $k = 0$), $a(t) = Ct^{2/3}$, so
$$d_h(t) = 3ct. \tag{9}$$

- So the sonic horizon distance is
$$d_s(t) = \sqrt{3}ct. \tag{10}$$
Using the time of decoupling $t_{\rm dec} \approx 3.8 \times 10^5$ yr,
$$d_s(t) = \sqrt{3}ct_{\rm dec} = 6.2 \times 10^{21}\ \text{m} = 200\ \text{kpc}, \tag{11}$$
comparable to the Hubble distance at the time of last scattering
$$d_H(z_{\rm ls}) = \frac{c}{H(z_{\rm ls})} \approx \frac{3.0\times 10^8\ \text{m s}^{-1}}{1.24\times 10^{-18}\ \text{s}^{-1}(1101)^{3/2}} \approx 6.6\times 10^{21}\ \text{m} \approx 0.2\ \text{Mpc} \tag{12}$$
where we have computed $H(z)$ at $z_{\rm ls} \approx 1100$ using the expression $H(z) = 1.24 \times 10^{-18}\ \text{s}^{-1}(1 + z)^{3/2}$.

- A patch of the last scattering surface with this physical size will have an angular size, as seen from Earth, of
$$\theta_s = \frac{d_s}{d_A} \approx \frac{0.2\ \text{Mpc}}{13\ \text{Mpc}} \approx 0.015\ \text{rad} \approx 1^\circ \tag{13}$$
We will see that the first peak in the $\Delta_T$ vs. $l$ curve corresponds to the sonic horizon $\theta_s$.

- Origin of fluctuations with $\theta > \theta_s$ ($l < 180$) is different from those with $\theta < \theta_s$ ($l > 180$) and we will consider them separately.

### 2.2 Sub-horizon scale fluctuations

- Fluctuations on scales smaller than the sonic horizon $\theta \leq \theta_s$ come from regions small enough that sound waves have had time to cross them by the time of decoupling. This means that the fluctuations depend on the behavior of photons and baryons.
- Immediately before decoupling, photons, electrons and protons make a single photon-baryon fluid. Its energy density is about a third of the dark matter, so it moves primarily under the gravitational influence of the dark matter rather than under its own self-gravity.
- If the photon-baryon fluid is in a potential well, it will fall to the center. If the size of the well is larger than the sonic horizon, the fluid, traveling at the sound speed $c_s < c$, will not have time to fall to the center by the time of last scattering. This is why the motions of photons and baryons don't matter on scales $\theta > \theta_s$ and why the fluctuations on these large scales are determined only by the dark matter distribution, as we will see later.
- On scales smaller than the horizon, $\theta < \theta_s$, oscillations develop. The falling photon-baryon fluid is compressed by gravity, and its pressure rises until it's sufficient to make the fluid expand outward. Expansion continues until pressure drops enough for gravity to cause it to fall inward again. This results in standing waves in the fluid called **acoustic oscillations**.
- The temperature is higher than average in regions of compression, and lower than average in regions of rarefaction. We see the imprint of these oscillations in the temperature fluctuations of the CMB.
- We can describe these oscillations with a very simple toy model. Consider a cylinder of cross-sectional area $A$ and length $2L$, filled with gas (see Figure 2). There is a movable piston in the center of the cylinder which represents the inertia of the baryons, so the piston has mass $m$ equal to the mass of baryons in the cylinder. The equilibrium values of the pressure and density are $P_0$ and $\rho_0$ respectively, so the mass of the piston is $m = 2LA\rho_0$.

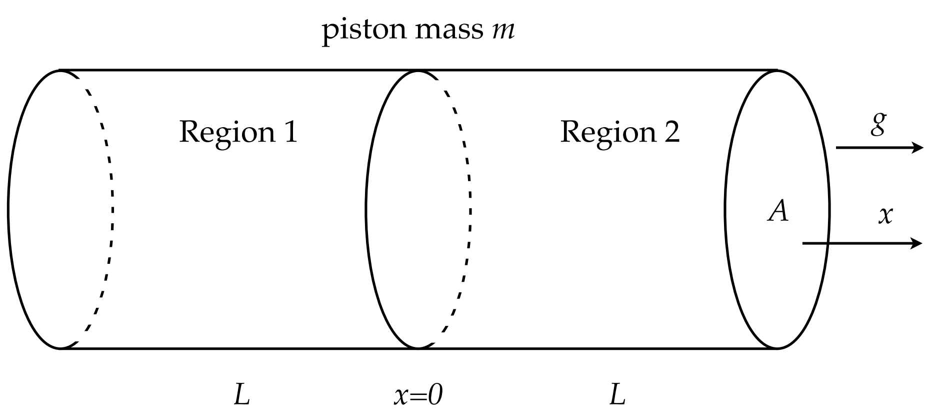


Figure 2: A cylinder of length $2L$ and cross-sectional area $A$ filled with gas. A moveable piston is in the center of the cylinder.

- If we displace the piston, the density on either side changes by a small amount $\Delta\rho$, which is accompanied by a pressure difference

$$\Delta P = P_1 - P_2 \tag{14}$$

where, to first order,

$$P_1 = P_0 + \frac{dP}{d\rho}\Delta\rho_1 \tag{15}$$

and

$$P_2 = P_0 + \frac{dP}{d\rho}\Delta\rho_2 \tag{16}$$

- Since the sound speed is

$$c_s = \left(\frac{\partial P}{\partial \rho}\right)^{1/2}, \tag{17}$$

we have

$$\Delta P = \frac{dP}{d\rho}(\Delta\rho_1 - \Delta\rho_2) = c_s^2(\Delta\rho_1 - \Delta\rho_2). \tag{18}$$

For a displacement $x$ of the piston,

$$\Delta\rho_1 = \rho_1 - \rho_0 = \rho_0\left(\frac{\rho_1}{\rho_0} - 1\right) = \rho_0\left(\frac{L}{L+x} - 1\right) \tag{19}$$

and

$$\Delta\rho_2 = \rho_2 - \rho_0 = \rho_0\left(\frac{\rho_2}{\rho_0} - 1\right) = \rho_0\left(\frac{L}{L-x} - 1\right) \tag{20}$$

since the mass $m = 2LA\rho_0$ of gas on either side doesn't change.

- So the pressure difference is

$$\Delta P = c_s^2\rho_0\left(\frac{1}{1+x/L} - \frac{1}{1-x/L}\right) \tag{21}$$

and to first order in $x$,

$$\Delta P = -2c_s^2\rho_0\left(\frac{x}{L}\right). \tag{22}$$

- Newton's second law for the piston of mass $m = 2LA\rho_0$ is

$$m\frac{d^2x}{dt^2} = A\Delta P \tag{23}$$

which is

$$2LA\rho_0\frac{d^2x}{dt^2} = -2c_s^2A\rho_0\left(\frac{x}{L}\right). \tag{24}$$

- The equation of motion is therefore

$$\frac{d^2x}{dt^2} = -\frac{c_s^2}{L^2}x, \tag{25}$$

which results in simple harmonic motion $x = x_0\sin(\omega t)$ with frequency

$$\omega = \frac{c_s}{L}. \tag{26}$$

This shows us that larger density fluctuations will oscillate more slowly, and that the frequency of oscillation depends on the sound speed, which depends on the baryon density and the equation of state.

- Now we add a uniform gravitational field of strength $g$, directed in the positive $x$ direction. This represents the effect of a concentration of dark matter on the baryons in the photon-baryon fluid. Because the dark matter isn't subject to the strong radiation pressure of the photons, it can form clumps that can gravitationally assist or resist the motion of the fluid.

- Newton's second law is now
$$m\frac{d^2x}{dt^2} = A\Delta P + mg \tag{27}$$
which gives
$$\frac{d^2x}{dt^2} = -\frac{c_s^2}{L^2}x + g. \tag{28}$$

- To solve this we define
$$y \equiv x - \frac{L^2}{c_s^2}g \tag{29}$$
so Newton's second law becomes
$$\frac{d^2y}{dt^2} = -\frac{c_s^2}{L^2}y. \tag{30}$$
As before the solution is simple harmonic motion with frequency $\omega = c_s/L$, but now the oscillations are about $y = 0$, corresponding to an equilibrium position
$$x_{\text{eq}} = \frac{L^2 g}{c_s^2} > 0. \tag{31}$$

- This is the behavior of a *forced harmonic oscillator*. The solution is shown in Figure 3. The solid line is the unforced solution, with $g = 0$: oscillations about the origin. The dashed curves are the forced solutions for two different frequencies. In both cases the zero point of the oscillations shifts in the direction of the force, and the effect is more dramatic for lower frequencies.

- The lower panel of Figure 3 shows the square of the oscillator position as a function of time. All three oscillators show a series of peaks at $t = n\pi/\omega$ corresponding to the maxima and minima. The odd modes are compression, and the even modes are rarefaction. In the case of the unforced oscillator, the heights of the peaks are identical, but in the forced case the heights of the odd peaks are greater than the heights of the even peaks, and the effect is most dramatic for low frequencies. This means that the compressions in the direction of the gravitational field are of greater magnitude than the rarefactions.

- This tells us that in the early universe, because of the gravitational effect of dark matter, collapsing fluctuations are favored over expanding ones. Because compressions are of greater magnitude than rarefactions, we expect the peaks of the angular power spectrum to be enhanced for odd harmonics (compressions) and diminished for even harmonics (rarefactions).

- The frequency depends on the baryon content of the universe, since baryons are heavy and reduce the sound speed. This means that both the position and relative heights of the additional peaks are very sensitive to $\Omega_b$ and to the relative amounts of dark and baryonic matter. Careful modeling of the CMB power spectrum is complex, but matching of these models to observations gives us precise measurements of $\Omega_0$, $\Omega_b$ and $\Omega_m$.

- Now we consider fluctuations on different scales at the time of recombination. Several modes of the acoustic oscillations are shown in Figure 4. The smaller a fluctuation is, the sooner it is included by the sound horizon and can begin oscillating. The first peak of the power spectrum is due to the compression of a large region that reached maximum compression at the time of decoupling. The

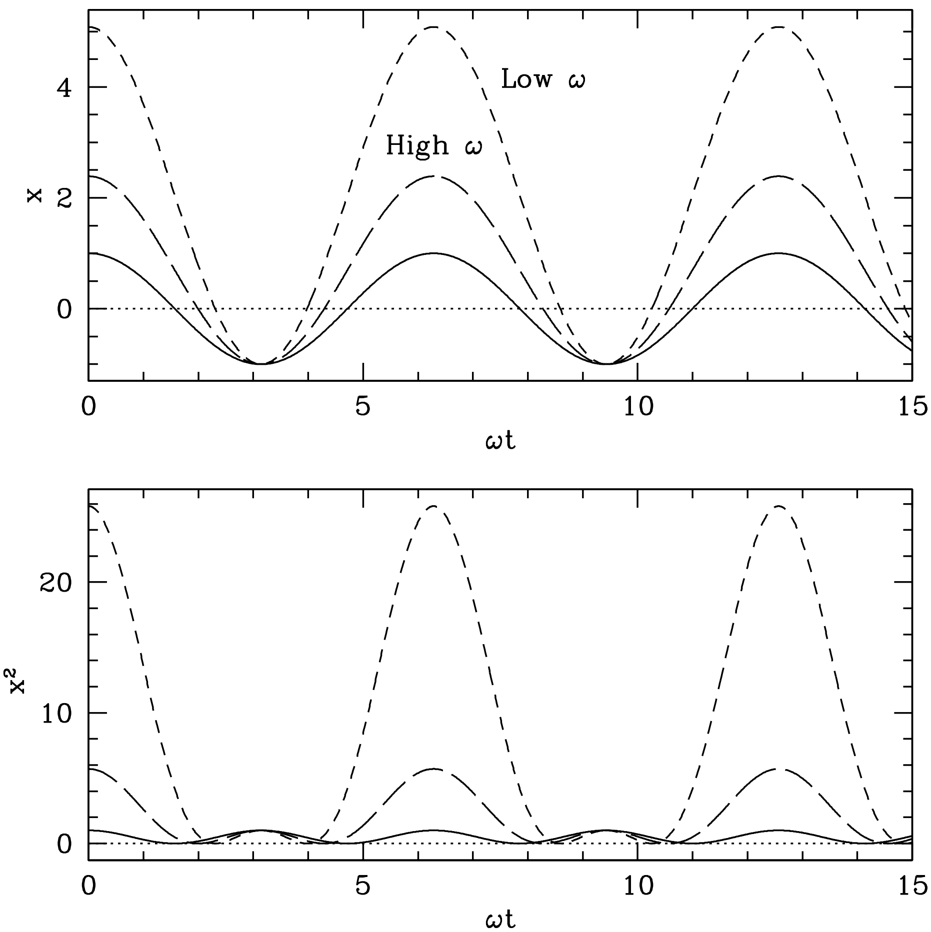


Figure 3: Displacement of a forced harmonic oscillator, for different frequencies.

first trough is produced by a smaller region that started oscillating earlier and is able to oscillate more quickly, so it arrives at $\delta T = 0$ at the time of decoupling. The second peak is due to the oscillation of a still smaller region that has passed through its maximum compression and reached maximum rarefaction at the time of decoupling. Note that the magnitude of $\delta T$ for the first peak is larger than that for the second peak, because of the biasing effect of dark matter discussed above.

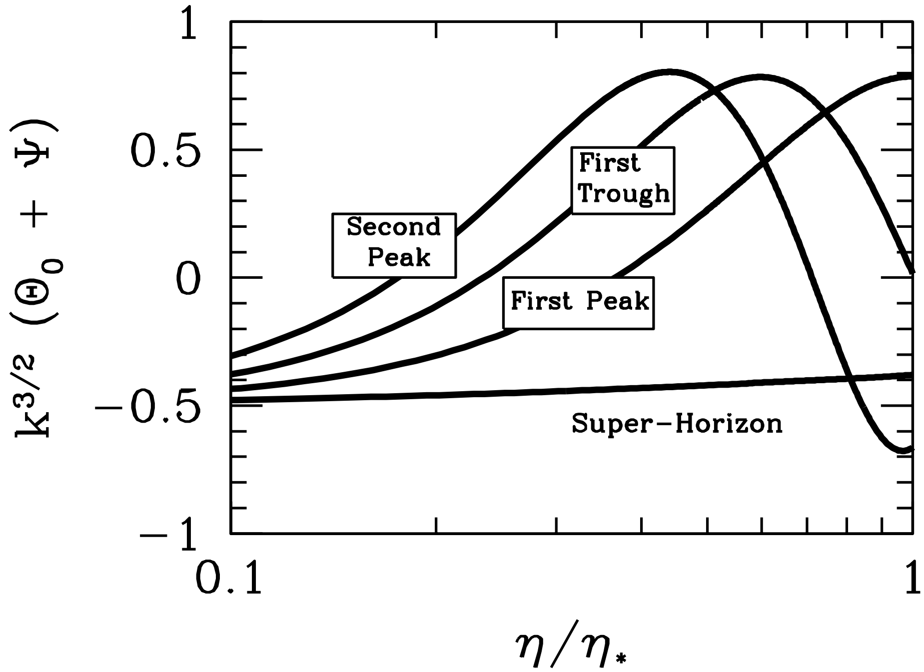


Figure 4: Several modes of acoustic oscillations in the early universe. The sum of potentials on the vertical axis can be thought of as $\delta T$, so positive values indicate compression of the photon-baryon fluid and negative values indicate rarefaction. The horizontal axis indicates time as a fraction of the time before recombination, so $\eta/\eta_* = 1$ at recombination. From Dodelson, *Modern Cosmology*.

- So, the first peak in the CMB power spectrum corresponds to maximum compression at the time of last scattering, and the size is equal to the sonic horizon at the time of last scattering. Higher $l$ peaks will be from higher harmonics.

- Location of first peak is a *standard ruler*, since it corresponds to the horizon size at the time of last scattering. It's sensitive to the cosmological parameters through the angular diameter distance; primarily sensitive to overall curvature. See Figures 6 and 5. The observed location of the first peak tells us that the universe is flat: $\Omega_0 = 1.02 \pm 0.02$ from WMAP.

- The suppression of the second peak increases as $\Omega_b$ increases, since a greater baryon density slows the oscillations. This means that the relative heights of the first two peaks gives $\Omega_b$. The third peak corresponds to a second maximum compression. This is sensitive to the density of dark matter, and the comparable heights of the second and third peaks tell us that most of the matter is dark.

- At even higher $l$, the peaks decrease. This is damping due to the finite distance a photon travels between scatterings. The mean free path between scatterings is

$$\lambda_{mfp} = \frac{1}{n_e \sigma_T}, \tag{32}$$

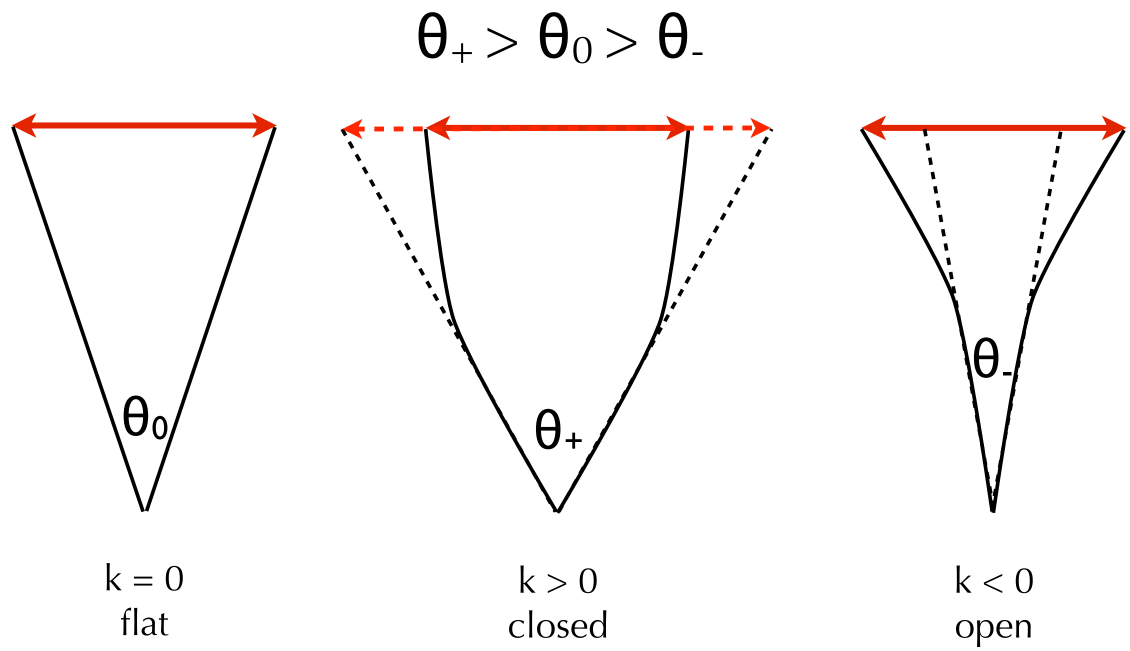


Figure 5: The angular size of the sound horizon at recombination depends on the curvature of the universe.

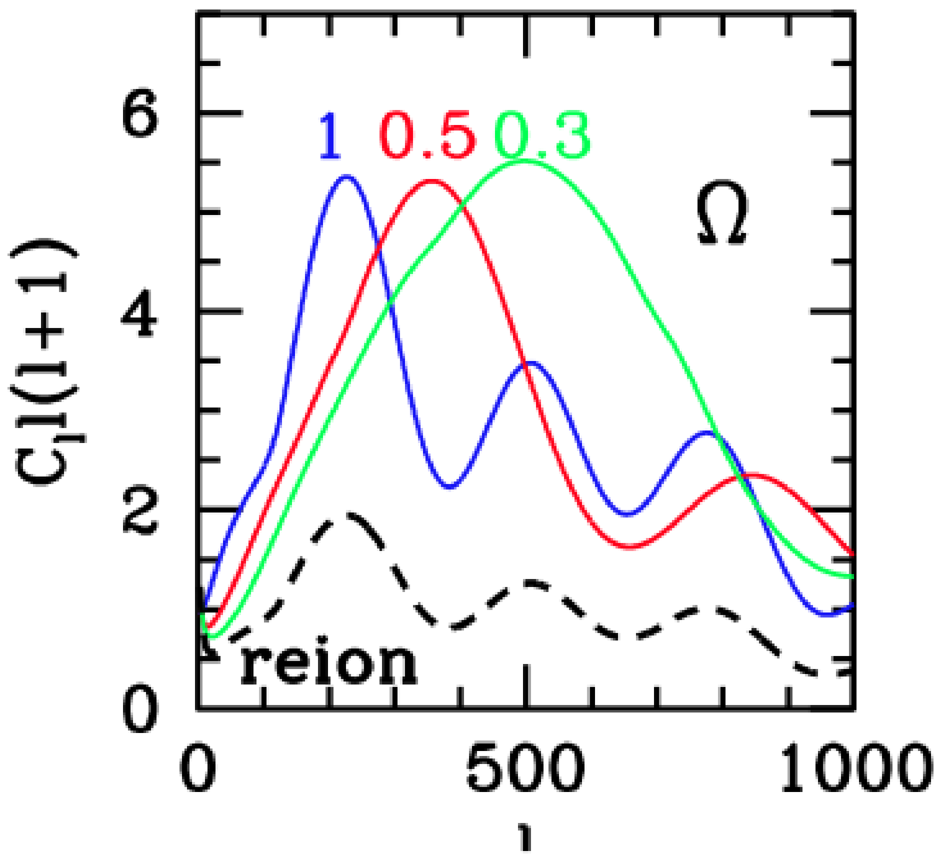


Figure 6: The position of the first peak in the CMB power spectrum depends on the curvature $\Omega$. From Kamionkowski & Kosowsky (1999).

so if the density of electrons is large the mean free path is small. In a Hubble time, a photon scatters $\sim n_e \sigma_T H^{-1} c$ times, and the total distance traveled is the mean free path times the square root of the total number of steps. So a photon moves a mean distance

$$\begin{aligned} \lambda_D &\sim \lambda_{mfp} \sqrt{n_e \sigma_T H^{-1} c} \\ &= \sqrt{\frac{c}{n_e \sigma_t H}} \end{aligned} \tag{33}$$

in a Hubble time. We expect perturbations on scales smaller than $\lambda_D$ to be washed out. Because $n_e$ is proportional to $\Omega_b$ when the universe is ionized, models with smaller baryon density have larger $\lambda_D$ and the damping sets in at larger scales. So the damping at high values of $l$ also tells us about $\Omega_b$. This is called **Silk damping**.

- Important to note that there are oscillations occurring on all scales, but we only see some of them, depending on their phase at the time of recombination.

- The existence of the oscillations in the cosmic microwave background provides strong evidence for inflation, since without inflation the perturbations on different scales wouldn't be in phase. The existence of the CMB harmonics shows us that the oscillations of all density fluctuations of a given size are *in phase*: they reach their maximum compressions and rarefactions at the same time, so they all began oscillating simultaneously.

- Inflation expands fluctuations to super-horizon size, where they remain frozen until the horizon encompasses them and allows them to oscillate. Without inflation, the fluctuations would be generated at random times and would not be in phase. This is an extension of the horizon problem: regions on the surface of last scattering not now in causal contact not only have the same temperature, but also oscillate in phase.