GUASA 2015: Cosmology
The growth of density perturbations

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The temperature anisotropies in the cosmic microwave background are of order $\delta T/T \sim 10^{-5}$, telling us that at $z \sim 1100$ the universe was extremely (but not perfectly) smooth. Today, however, the universe looks like Figure **??**. The distribution of galaxies is inhomogeneous, characterized by the superclusters and voids that are the most prominent features in Figure **??**. Superclusters are generally roughly planar or linear structures that are in the process of collapsing due to their own self-gravity. They contain one or more clusters of galaxies, which are the largest fully collapsed structures (meaning that they have come into equilibrium and obey the virial theorem). Voids are underdense regions roughly spherical in shape. The development of this irregular matter distribution is known as **structure formation**.

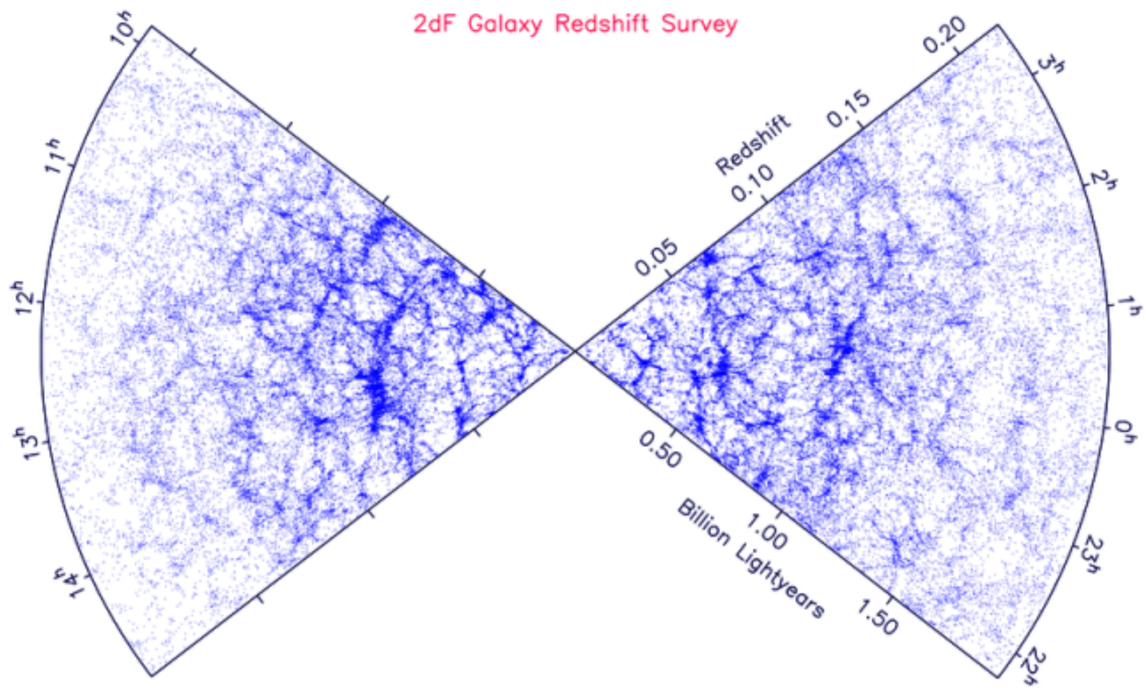

Figure 1: The distribution of galaxies on large scales, from the 2dF (2 degree Field) redshift survey.

How did we get from the very smooth distribution of matter in the early universe to the clumpy distribution we see today? The answer is **gravitational instability**: small, overdense regions expand less rapidly than the universe as a whole, and will, if their density is high enough, collapse and become gravitationally bound objects such as galaxy clusters. These dense regions will also draw matter from underdense regions, further enhancing the density contrast.

1 The growth of density fluctuations

To see how this works, we will consider a component of the universe whose energy density is a function of both position and time, $u(\vec{r}, t)$. At a given time t, the spatially averaged energy density is

$$\bar{u}(t) = \frac{1}{V}\int_V u(\vec{r}, t) d^3r, \tag{1}$$

where we need to be sure that we are averaging over a volume V that is large compared to the biggest structures in the universe.

We will define a dimensionless density fluctuation

$$\delta(\vec{r},t) \equiv \frac{u(\vec{r},t) - \bar{u}(t)}{\bar{u}(t)}. \tag{2}$$

This is the difference in density of a region of interest relative to the average density. δ is negative in underdense regions and positive in overdense regions, with a minimum value $\delta = -1$ corresponding to $u = 0$.

We want to know how a small fluctuation in density, with $|\delta| \ll 1$, grows in amplitude under the influence of gravity. When $|\delta| \ll 1$, we can address this with linear perturbation theory.

We will consider a simple case of a static, homogeneous, matter-only universe with uniform mass density $\bar{\rho}$ (such a static, matter-only universe can't exist, which is why Einstein introduced the cosmological constant, but for the moment we will consider it approximately static and address the effects of expansion later). We add a small amount of mass to a spherical region with radius R, so that the density in the sphere is $\rho = \bar{\rho}(1+\delta)$, with $\delta \ll 1$. If the density excess δ is uniform throughout the sphere, then the gravitational acceleration at the sphere's surface due to the excess mass will be

$$\ddot{R} = -\frac{G(\Delta M)}{R^2} = -\frac{G}{R^2}\left(\frac{4\pi}{3}R^3\bar{\rho}\delta\right) \tag{3}$$

or

$$\frac{\ddot{R}}{R} = -\frac{4\pi G\bar{\rho}}{3}\delta(t). \tag{4}$$

So a mass excess with $\delta > 0$ will cause the sphere to collapse inward ($\ddot{R} < 0$).

We also know that the mass of the sphere

$$M = \frac{4\pi}{3}\bar{\rho}[1+\delta(t)]R(t)^3 \tag{5}$$

remains constant during the collapse. We can rewrite this as

$$R(t) = R_0[1+\delta(t)]^{-1/3} \tag{6}$$

where

$$R_0 \equiv \left(\frac{3M}{4\pi\bar{\rho}}\right)^{1/3} = \text{ constant.} \tag{7}$$

If $\delta \ll 1$ we can make the approximation

$$R(t) \approx R_0\left[1 - \frac{1}{3}\delta(t)\right]. \tag{8}$$

We take the second time derivative to get

$$\ddot{R} \approx -\frac{1}{3}R_0\ddot{\delta} \approx -\frac{1}{3}R\ddot{\delta}, \tag{9}$$

so, from mass conservation,

$$\frac{\ddot{R}}{R} \approx -\frac{1}{3}\ddot{\delta} \tag{10}$$

in the limit that $\delta \ll 1$.

Combining this with Equation **??**, we get an expression for the evolution of the overdensity δ:

$$\ddot{\delta} = 4\pi G\bar{\rho}\delta. \tag{11}$$

The general solution of Equation **??** is

$$\delta(t) = A_1 e^{t/t_{\rm dyn}} + A_2 e^{-t/t_{\rm dyn}}, \tag{12}$$

where the dynamical time for collapse is

$$t_{\rm dyn} = \frac{1}{(4\pi G\bar{\rho})^{1/2}} = \left(\frac{c^2}{4\pi G\bar{u}}\right)^{1/2}. \tag{13}$$

Note that the dynamical time depends on $\bar{\rho}$ but not on the size of the overdensity R. The constants A_1 and A_2 depend on the initial conditions; if the overdense sphere starts at rest with $\dot{\delta} = 0$ at $t = 0$, then $A_1 = A_2 = \delta(0)/2$.

After a few dynamical times, only the exponentially growing solution is significant, showing that gravity tends to make small density fluctuations in a static, pressureless medium grow exponentially with time.

2 The Jeans length

The dynamical time for air of density $\bar{\rho} \approx 1$ kg m^{-3} is about 9 hours, but clearly the air around us is in no danger of exponential collapse. This is because of pressure support. A nonrelativistic gas obeys the ideal gas law

$$P = \frac{kT}{\mu c^2} u, \tag{14}$$

where μ is the mean mass of the gas particles and we write it in terms of the energy density $u = \rho c^2$. The equation of state parameter is therefore

$$w \approx \frac{kT}{\mu c^2}. \tag{15}$$

The pressure of an ideal gas is zero only at zero temperature.

As a sphere of gas is compressed by its own gravity, a pressure gradient builds up that tends to counter the force of gravity. A star is the prime example of an object in hydrostatic equilibrium, where gravity is balanced by a pressure gradient. It isn't always possible to attain hydrostatic equilibrium, however.

Consider our overdense sphere with initial radius R. Without pressure, it would collapse on a timescale

$$t_{\rm dyn} \sim \frac{1}{(G\bar{\rho})^{1/2}} \sim \left(\frac{c^2}{G\bar{u}}\right)^{1/2}. \tag{16}$$

If the pressure is nonzero, the collapse will be countered by a pressure gradient. The pressure gradient can't develop instantaneously, however, because pressure changes travel at the sound speed. So the time it takes the pressure gradient to build up in a region of radius R will be

$$t_p \sim \frac{R}{c_s}, \tag{17}$$

where c_s is the local sound speed. In a medium with equation of state parameter $w > 0$, the sound speed is

$$c_s = c\left(\frac{dP}{du}\right)^{1/2} = \sqrt{w}c. \tag{18}$$

In order to attain hydrostatic equilibrium, the pressure gradient must build up before the overdense region collapses, requiring

$$t_p < t_{\rm dyn}. \tag{19}$$

Comparing the dynamical time in Equation **??** with the pressure timescale in Equation **??**, we find that for a density perturbation to be stabilized by pressure against collapse, it must be smaller than a size given by the relation

$$\lambda_J \sim c_s t_{\rm dyn} \sim c_s\left(\frac{c^2}{G\bar{u}}\right)^{1/2}. \tag{20}$$

This length scale is known as the **Jeans length**. Overdense regions larger than the Jeans length collapse under their own gravity, while overdense regions smaller than the Jeans length oscillate in density as stable sound waves.

A more precise derivation of the Jeans length, including the numerical factors, gives

$$\lambda_J = c_s\left(\frac{\pi c^2}{G\bar{u}}\right)^{1/2} = 2\pi c_s t_{\rm dyn}. \tag{21}$$

The Jeans length of the Earth's atmosphere, where $c_s \approx 0.3$ km s^{-1} and the dynamical time is 9 hours, is $\lambda_J \sim 10^5$ km, much longer than the scale height of the atmosphere. So there is no danger of density fluctuations in the air collapsing.

The Jeans length is an important concept in astrophysics, with other applications. As we'll see later, we often write it in terms of the mass that can collapse rather than the length scale; this is the **Jeans mass**. As as aside, consider a cloud of gas in a galaxy, which may or may not collapse to form stars. In order for the cloud to collapse, we require the gravitational potential energy to be greater than twice the kinetic energy, $2K < |U|$. We can easily derive an approximate expression for the Jeans mass. The potential energy of our gas cloud is

$$U = -\frac{3}{5}\frac{GM^2}{R}, \tag{22}$$

and we'll treat it as an ideal gas to find its kinetic energy

$$K = \frac{3}{2}NkT \tag{23}$$

where N is the number of particles. We write N in terms of the cloud mass and the mean molecular weight μ,

$$N = \frac{M}{\mu m_H}. \tag{24}$$

We also write the radius R in terms of the cloud mass and density

$$R = \left(\frac{3M}{4\pi\rho}\right)^{1/3}. \tag{25}$$

Then the condition for collapse, $2K < |U|$, requires

$$M > M_J \simeq \left(\frac{5kT}{G\mu m_H}\right)^{3/2}\left(\frac{3}{4\pi\rho}\right)^{1/2} \tag{26}$$

where M_J is the Jeans mass. So, as we might expect intuitively, a cloud of mass M becomes less likely to collapse as its temperature increases, and more likely to collapse as its density increases. This is important to the process of star formation, but we'll return to it in the context of perturbations in the early universe.

To consider the behavior of density fluctuations on cosmological scales, consider a flat universe with mean density $\bar{u}$, containing density fluctuations with amplitude $|\delta| \ll 1$. The characteristic time for the expansion of this universe is the Hubble time

$$H^{-1} = \left(\frac{3c^2}{8\pi G\bar{u}}\right)^{1/2}. \tag{27}$$

Comparison of this timescale with the dynamical time in Equation **??** shows that the timescales are comparable:

$$H^{-1} = \left(\frac{3}{2}\right)^{1/2} t_{\rm dyn} \approx 1.22\, t_{\rm dyn}. \tag{28}$$

The Jeans length in an expanding flat universe will then be

$$\lambda_J = 2\pi c_s t_{\rm dyn} = 2\pi \left(\frac{2}{3}\right)^{1/2} \frac{c_s}{H}. \tag{29}$$

If we focus on one particular component of the universe with equation of state parameter w and sound speed $c_s = \sqrt{w}c$, the Jeans length for that component is

$$\lambda_J = 2\pi \left(\frac{2}{3}\right)^{1/2} \sqrt{w}\frac{c}{H}. \tag{30}$$

For example, the radiation component of the universe has $w = 1/3$, so the sound speed in a gas of photons or other relativistic particles is

$$c_s = c/\sqrt{3} \approx 0.58c. \tag{31}$$

The Jeans length for radiation in an expanding universe is then

$$\lambda_J = \frac{2\pi\sqrt{2}}{3}\frac{c}{H} \approx 3.0\frac{c}{H}. \tag{32}$$

Density fluctuations in the radiative component will be pressure-supported if they are smaller than three times the Hubble distance. A universe containing only radiation can have density perturbations smaller than $\lambda_J \sim 3c/H$, but they will be stable sound waves and will not collapse under their own gravity.

In order for a universe to have gravitationally collapsed structures much smaller than the Hubble distance, it must have a non-relativistic component, with $\sqrt{w} \ll 1$.

The gravitational collapse of the *baryonic* component of the universe is complicated by the fact that it was coupled to the photons until decoupling at $z_{\rm dec} \approx 1100$. We saw earlier that the Hubble distance at the time of last scattering (effectively the same as the time of decoupling) was $c/H(z_{\rm dec}) \approx 0.2$ Mpc. At this time, the energy density of baryons was

$$u_b \approx 2.8 \times 10^{11} \text{ MeV m}^{-3}, \tag{33}$$

corresponding to a mass density

$$\rho_b \approx 5.0 \times 10^{-19} \text{ kg m}^{-3}, \tag{34}$$

and the energy density of photons was

$$u_\gamma \approx 3.8 \times 10^{11} \text{ MeV m}^{-3} \approx 1.4 u_b. \tag{35}$$

Before decoupling, the photons, electrons and baryons were coupled into a single fluid. Since the photons dominated over the baryons, with $u_\gamma > u_b$, we will consider the baryons to be dynamically insignificant. Just *before* decoupling, the Jeans length of the photon-baryon fluid was then roughly the same as the Jeans length of a pure photon gas,

$$\lambda_J \approx 3c/H(z_{\rm dec}) \approx 0.6 \text{ Mpc} \approx 1.9 \times 10^{22} \text{ m}. \tag{36}$$

The **baryonic Jeans mass** M_J (also discussed above) is defined as the mass of baryons contained within a sphere of radius λ_J:

$$M_J \equiv \rho_b \left(\frac{4\pi}{3} \lambda_J^3 \right). \tag{37}$$

Immediately before decoupling, the baryonic Jeans mass was

$$\begin{aligned} M_J &\approx 5.0 \times 10^{-19} \text{ kg m}^{-3} \left(\frac{4\pi}{3} \right) (1.9 \times 10^{22} \text{ m})^3 & (38)\\ &\approx 1.3 \times 10^{49} \text{ kg} & (39)\\ &\approx 7 \times 10^{18} \text{ M}_\odot. & (40) \end{aligned}$$

This is about 3×10^4 times greater than the estimated baryonic mass of the Coma cluster, and is larger than the baryonic mass of the largest superclusters seen today.

Now we will consider the effect of decoupling on the baryonic Jeans mass. Once the photons are decoupled, the photos and baryons form two separate gases instead of a single fluid. The sound speed of the photon gas is

$$c_s(\text{photon}) = c/\sqrt{3} \approx 0.58c, \tag{41}$$

while the sound speed in the baryonic gas is

$$c_s(\text{baryon}) = \left(\frac{kT}{mc^2} \right)^{1/2} c, \tag{42}$$

from the equation of state parameter for an ideal gas given in Equation **??**.

At the time of decoupling, the thermal energy per particle was $kT_{\rm dec} \approx 0.26$ eV, and the mean rest energy of the atoms in the baryonic gas was $mc^2 = 1.22 m_p c^2 \approx 1140$ MeV, accounting for the helium mass fraction of $Y_p = 0.24$. So the sound speed of the baryonic gas immediately after decoupling was

$$c_s(\text{baryon}) \approx 1.5 \times 10^{-5} c, \tag{43}$$

putting numbers into Equation **??** above. This is only 5 km s^{-1}. So once the baryons decoupled from the photons, their associated Jeans length decreased by a factor

$$F = \frac{c_s(\text{baryon})}{c_s(\text{photon})} \approx \frac{1.5 \times 10^{-5}}{0.58} \approx 2.6 \times 10^{-5}. \tag{44}$$

Decoupling causes the baryonic Jeans mass to decrease by a factor $F^3 \approx 1.8 \times 10^{-14}$, so it drops drastically from $M_J(\text{before}) \approx 7 \times 10^{18}$ M$_\odot$ to

$$M_J(\text{after}) = F^3 M_J(\text{before}) \approx 1 \times 10^5 \text{ M}_\odot. \tag{45}$$

This is comparable to the baryonic mass of the smallest known dwarf galaxies, and is much smaller than the baryonic mass of the Milky Way, which is $\sim 10^{11}$ M$_\odot$.

This abrupt decrease in the baryonic Jeans mass at the time of decoupling is a key event in the formation of structure in the universe. Perturbations in the baryon density, whether on the scales of superclusters or dwarf galaxies, couldn't grow until the universe was $t_{\rm dec} \approx 0.35$ Myr old. After decoupling, the baryonic density perturbations could collapse and grow.