GUASA 2015: Cosmology
The power spectrum

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1 The power spectrum

1.1 The horizon

Linear perturbation theory shows that fluctuations grow independently of each other on all scales, or for all wave numbers. This is true in the framework of general relativity as well as in the Newtonian case, as long as the fluctuation amplitudes are small. At very early times, fluctuations with a comoving length scale λ may be larger than the comoving horizon

$$d_{\mathrm{h,c}}(t) = \int_0^t \frac{c\,dt}{a(t)}, \tag{1}$$

where the c subscript indicates that this is the *comoving* horizon distance. Note that this differs by a factor of $a(t)$ from the *proper* horizon distance we calculated in Lecture 8.

We define $z_{\mathrm{ent}}(\lambda)$ as the redshift at which the comoving horizon is equal to the comoving length scale λ,

$$d_{\mathrm{h,c}}(z_{\mathrm{ent}}(\lambda)) = \lambda. \tag{2}$$

Only for redshifts $z < z_{\mathrm{ent}}(\lambda)$ does the horizon become larger than the scale under consideration. It is common to say that at $z_{\mathrm{ent}}(\lambda)$ the perturbation under consideration "enters the horizon," but in fact the process is the opposite: the horizon outgrows the perturbation.

Relativistic perturbation theory shows that density fluctuations of scale λ grow as long as $\lambda > d_{\mathrm{h,c}}$, as a^2 if radiation dominates ($z > z_{\mathrm{eq}}$) or as a if matter dominates ($z < z_{\mathrm{eq}}$). This is because free-streaming particles or pressure gradients can't impede growth on scales larger than the horizon length, since physical interactions cannot extend to scales larger than the horizon size.

This has important effects on the transfer function, which we will consider qualitatively. If a perturbation enters the horizon in the radiation-dominated phase, i.e. $z_{\mathrm{ent}} > z_{\mathrm{eq}}$, it cannot grow until $z < z_{\mathrm{eq}}$ because, as we have seen, the expansion rate in the radiation era prohibits efficient growth of perturbations. If the perturbation enters the horizon during the matter-dominated epoch, i.e. $z_{\mathrm{ent}} < z_{\mathrm{eq}}$, it will grow as described earlier, with $\delta \propto D_+(t)$.

This means that a length scale λ_{eq} is singled out, for which

$$z_{\mathrm{eq}} = z_{\mathrm{ent}}(\lambda_{\mathrm{eq}}). \tag{3}$$

In other words, λ_{eq} is the comoving horizon size at equality, which is

$$\lambda_{\mathrm{eq}} = d_{\mathrm{h,c}}(z_{\mathrm{eq}}) \simeq 12\,(\Omega_{m,0}h^2)^{-1}\ \mathrm{Mpc}. \tag{4}$$

Density fluctuations with $\lambda > \lambda_{\mathrm{eq}}$ enter the horizon during the matter-dominated era, and therefore their growth is not impeded by a phase of radiation dominance. In contrast, density fluctuations with $\lambda < \lambda_{\mathrm{eq}}$ enter the horizon when radiation dominates. They will not grow significantly as long as $z > z_{\mathrm{eq}}$; only once the universe becomes matter-dominated will they start to grow again. This means that by the present time they have grown by a smaller factor than the perturbations with $\lambda > \lambda_{\rm eq}$.

To summarize, perturbations which enter the horizon during the radiation era grow as

$$\delta = \begin{cases} a^2 & z > z_{\rm ent} \\ \ln a & z_{\rm eq} < z < z_{\rm ent} \\ a & z < z_{\rm eq}. \end{cases} \tag{5}$$

This means that relative to a perturbation that enters the horizon during the matter-dominated epoch, the smaller perturbation is suppressed by a factor $\sim (a_{\rm eq}/a_{\rm ent})^2$, where we have neglected the logarithmic growth for $z_{\rm eq} < z < z_{\rm ent}$. This is shown in Figure **??**.

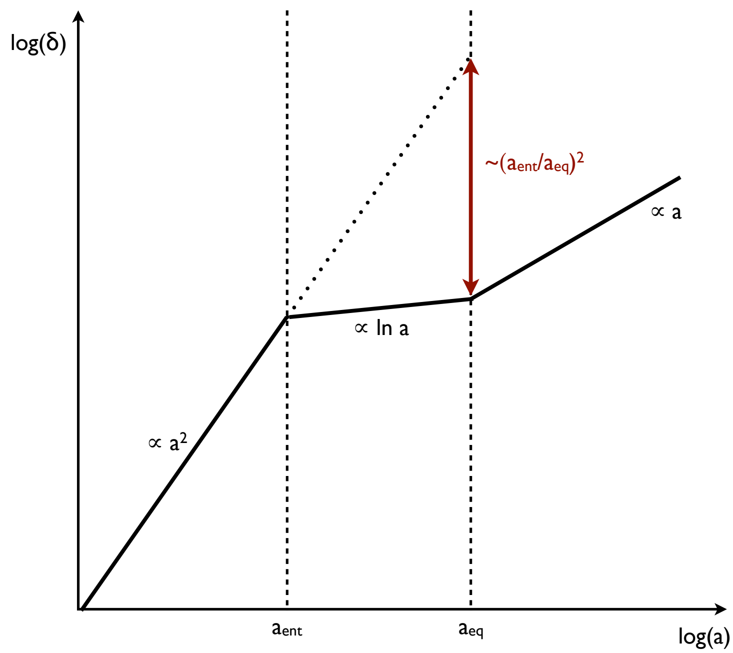

Figure 1: The growth of perturbations that enter the horizon during the radiation era is suppressed by a factor $\sim (a_{\rm eq}/a_{\rm ent})^2$.

1.2 The transfer function

Recall that the transfer function $T(k)$ modifies the power spectrum:

$$P(k) \propto k^n \, T^2(k). \tag{6}$$

The quantitative calculation of the effects discussed above allows us to compute the transfer function. In practice this is complicated and so is done numerically, but two limiting cases can be treated analytically:

$$T(k) \approx 1 \text{ for } k \ll k_{\rm eq} = 1/\lambda_{\rm eq} \tag{7}$$
$$T(k) \approx (k\lambda_{\rm eq})^{-2} \text{ for } k \gg k_{\rm eq} = 1/\lambda_{\rm eq}. \tag{8}$$

Perturbations with $k \ll k_{\rm eq}$ ($\lambda \gg \lambda_{\rm eq}$) are able to grow during the entire radiation era, because they are outside the horizon. Therefore they retain their original spectrum, and $T(k) \approx 1$. Perturbations with

$k \gg k_{\rm eq}$ enter the horizon during the radiation era and are suppressed until matter domination. $P(k)$ is therefore suppressed for $k \gg k_{\rm eq}$, with the largest suppression occurring for the largest wavenumbers, which enter the horizon at earlier times.

This theory predicts the shape of the power spectrum, but observations are necessary to measure its amplitude. A compilation of measurements of the present day power spectrum is shown in Figure **??**. For small k the initial power spectrum is unmodified: $P(k) \propto P_0(k) \propto k^n$ with $n \approx 1$. For large k, $P(k) = P_0(k)T(k)^2 \propto k^n k^{-4}$, so the power spectrum is strongly suppressed.

As we saw in Section 1.1 of Lecture 18, the root mean square mass fluctuations $\delta M/M \propto (k^3 P(k))^{1/2} \propto M^{-(n+3)/6}$. The suppression of the power spectrum for large k means that mass fluctuations are largest in amplitude for the smallest mass scales. This implies that in a universe filled with cold dark matter, the first objects to form are the smallest, with galaxies forming first, then clusters, then superclusters. This is known as the *bottom-up* theory of galaxy formation, and is consistent with our observation that galaxies have $\delta \gg 1$ while superclusters have $\delta \sim 1$ and are just beginning to collapse.

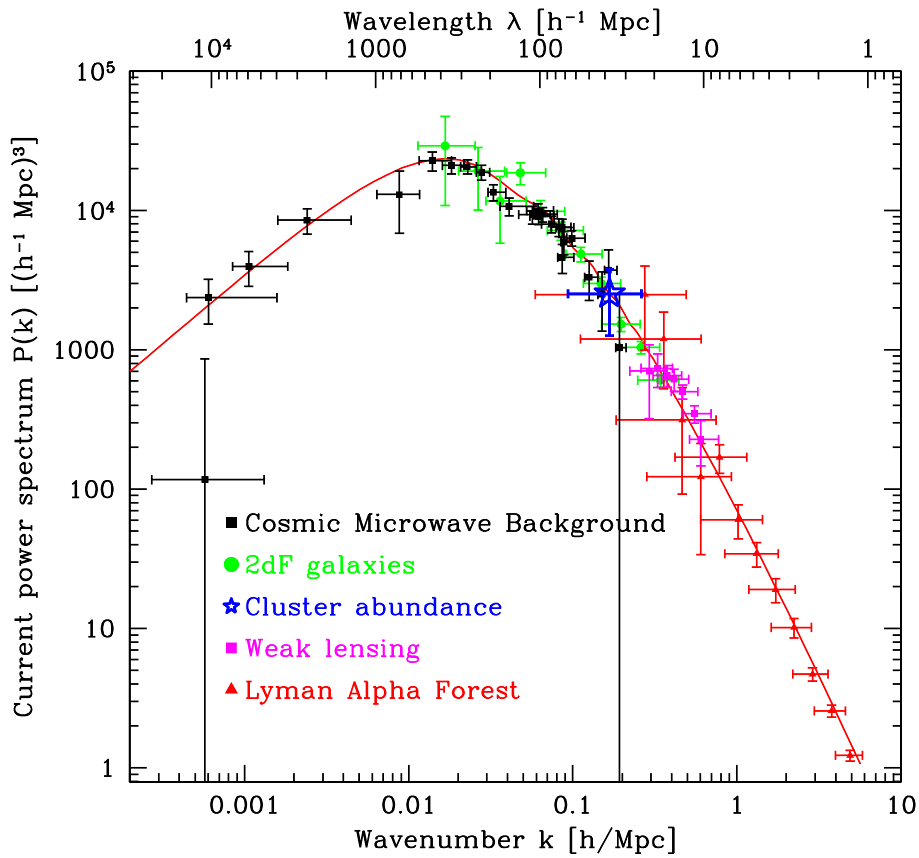

Figure 2: Measurements of the present day linear matter power spectrum $P(k)$. From Tegmark & Zaldarriaga 2002, *Phys Rev D* 66, 10.

2 Complications: redshift space distortions and bias

Figure **??** shows that the matter power spectrum is measured in several ways: CMB anisotropies on the largest scales (recall that the Sachs-Wolfe effect is due to fluctuations in the dark matter distribution on large scales), galaxy surveys in the middle of the range, and weak lensing and the Lyα forest on smaller scales. We will discuss weak lensing and the Lyα forest later; for now, we will describe some of the complications involved in using galaxy surveys to determine the power spectrum.

In order to interpret the observed 3D distribution of galaxies in terms of the underlying power spectrum of the matter distribution, we need to take into account two complications: peculiar velocities (relative to the Hubble flow) which affect our determination of distance from the measured galaxy redshifts, and the fact that light (i.e. galaxies) may not be an unbiased tracer of the mass.

This means that even if we measure accurately the redshifts of many galaxies in a region on the sky, the result is not a true 3D picture. This is because we do not observe galaxies in 3D. Rather, we observe their angular position on the sky θ, and redshift z (at the distances of interest there is no z-independent distance estimator). But redshift has two components: the cosmological component due to the expansion of the universe, and the Doppler effect of peculiar velocities:

$$1 + z_{\rm tot} = (1 + z_{\rm cosm})(1 + z_{\rm kin}), \tag{9}$$

or

$$z = \frac{dH_o + v}{c}. \tag{10}$$

We can thus define a redshift space **s** which is a transform of the real (or proper) space **r**, as follows:

$$\begin{aligned} s_1 &= r_1 = \frac{zc}{H_0}\theta_1 && (11)\\ s_2 &= r_2 = \frac{zc}{H_0}\theta_2 && (12)\\ s_3 &= r_3 + \frac{v_3}{H_0}. && (13) \end{aligned}$$

It is the radial axis of redshift which is modified by the Doppler effects of peculiar velocities. The complication is that the peculiar velocities arise from the clustering itself. Thus, the apparent clustering pattern in redshift space differs systematically from that in real space and the spatial correlation function of galaxies, $\xi_g(r)$, which is isotropic in real space is no longer isotropic in redshift space.

There are two effects at work here. The first, termed the "Fingers of God," is due to the velocity dispersion of galaxies within rich clusters and stretches out a cluster in redshift space. Since this affects only redshift and not position on the sky, the stretching occurs only radially (this is why the "fingers" point back to the observer). The other important redshift distortion is the **Kaiser effect**, due to galaxies bound to a central mass and still undergoing infall. It differs from the Fingers-of-God in that the peculiar velocities are coherent, not random, towards the central mass, though the effect is more subtle. The two effects are sketched in Figure **??**, and the Finger of God effect on galaxy redshift surveys is shown in Figure **??**.

It can be shown that the redshift-space and real-space density fields are related via

$$\delta_{\rm m,z} = \delta_{\rm m,r}[1 + f(\Omega_{m,0})\mu^2] \tag{14}$$

where μ is the cosine of the angle between the velocity vector and the line of sight, and $f(\Omega_{m,0}) \approx \Omega_{m,0}^{0.6}$ is the "velocity suppression factor" which increases for larger mass densities.

The second complication arises from the fact that in galaxy redshift surveys we map out the distribution of the light, whereas we are interested in the mass density field, not necessarily the same thing. The relation between mass and light is determined by complex physical processes, but is generally described by a linear

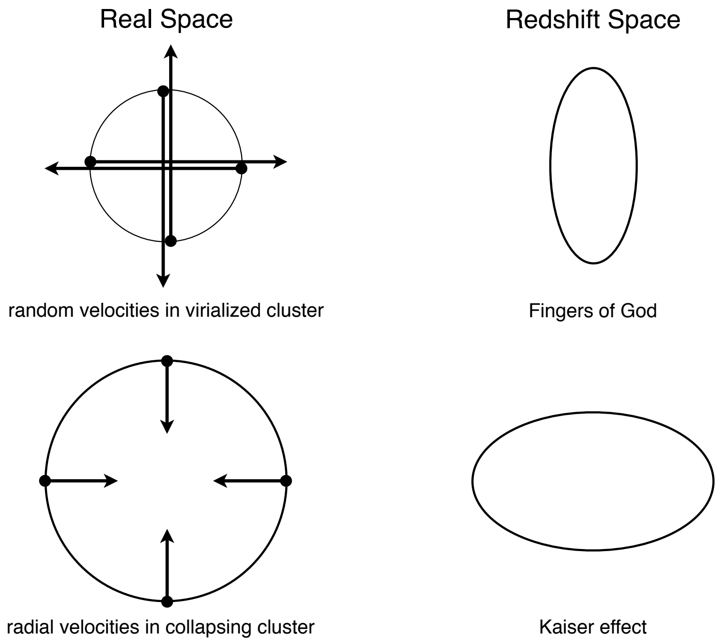

Figure 3: Redshift-space distortions due to line of sight velocities in galaxy clusters.

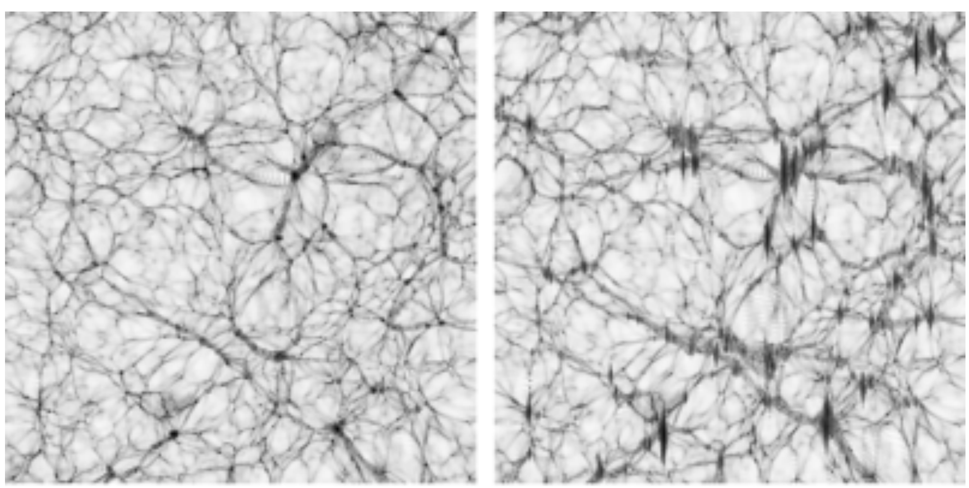

Figure 4: The "finger of god" effect in a simulated redshift survey. The left panel shows the true density distribution, and the right panel shows the redshift distortions when the distribution is observed from a point below the panel. The galaxy clusters appear as fingers pointing toward the observer because the random line-of-sight velocities in the clusters smear out distances derived from the Hubble law.

bias parameter b (that is, we assume some linear response of the galaxy formation process to small density perturbations), such that:

$$\delta_{\rm lum} = b\,\delta_m = \delta_m + (b-1)\delta_m \tag{15}$$

The point of the trivial rearrangement is to emphasize that the observed density fluctuation is a mixture of the dynamically generated density fluctuation, plus an additional term due to bias, which populates different regions of space in different ways. The first term is associated with peculiar velocities, but the second is not—the enhancements in galaxy densities are just some additional pattern.

In redshift space, we therefore add the anisotropic perturbations due to the dynamical component to the isotropic biased component, to obtain

$$\delta_{\rm lum,z} = \delta_{\rm m,r}[1 + f(\Omega_{m,0})\mu^2] + (b-1)\delta_{\rm m,r} = \delta_{\rm lum,r}\left[1 + \frac{f(\Omega_{m,0})\mu^2}{b}\right]. \tag{16}$$

Redshift-space effects thus give us a characteristic anisotropy of clustering, which can be used to measure the parameter $\beta = \Omega_{m,0}^{0.6}/b$. The power spectra in redshift and real space are related by

$$\frac{P_z}{P_r} = (1 + \beta\mu^2)^2. \tag{17}$$